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A novel amplification-fitted and phase controlled 6-point block method for second order oscillatory problems

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Abstract This study presents a family of non-symmetric, explicit six-point block methods of sixth order for solving second-order ordinary and partial differential equations. The construction focuses on reducing phase and amplification errors to improve performance in oscillatory and dissipative problems. The order, phase-lag behavior, and stability properties of the methods are analyzed, and their stability regions are illustrated. Numerical experiments on benchmark ODE and PDE problems confirm that the proposed schemes deliver higher accuracy. Comparisons with 12-step explicit symplectic RungeKutta methods and trigonometric-fitted schemes further show that the new methods are computationally faster.

Keywords Amplification-fitted method · Initial value problems (IVPs) · Block methods · Phase-fitting

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1 Introduction

Second-order differential equations with oscillatory solutions arise frequently in real-world phenomena such as mechanical vibrations, electrical circuits, orbital motion, and wave propagation. The accurate and efficient numerical solution

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of such equations is crucial, as analytical solutions are rarely obtainable for most practical problems.

In this study, we consider differential equations of the form

$$\frac{d^2w}{dx^2} = g(x, w), \quad w(x_0) = w_0,\ w'(x_0) = w_1,\ x \in [x_0, x] \tag{1}$$

which exclude the first derivative term w' such that the equation has unique solution. Such equations often represent conservative systems where the total energy remains constant, making phase accuracy and stability essential aspects of numerical methods. Over the years, various numerical approaches have been developed for solving this class of problems, including Runge-Kutta methods, methods designed to minimize phase and amplification errors, as well as exponentially fitted and trigonometrically fitted methods [1–6]. While Runge–Kutta methods are general-purpose and widely used, they tend to perform poorly for oscillatory problems due to their inability to accurately capture the phase and amplitude of oscillations over long integration intervals.

To achieve better accuracy in oscillatory systems, it is important to consider both phase error (which affects the timing of oscillations) and amplification error (which affects their magnitude). Minimizing these errors leads to methods that preserve the true qualitative behavior of oscillatory solutions. Furthermore, to enhance computational efficiency, block methods based on the concept of parallel computation offer significant advantages. These methods allow the simultaneous computation of multiple solution points within a block, and unlike one-step methods, they do not require transforming the second-order differential equation into a system of first-order equations, thereby reducing computational cost and complexity. Over the years, symmetric multi-step methods have attracted significant attention for solving oscillatory and time-reversible problems, owing to their high phase accuracy, energy-conserving nature, and excellent long-term stability. The pioneering work of Lambert and Watson [7], followed by Quinlan and Tremaine [8], demonstrated the superior performance of high-order symmetric schemes in simulating planetary motions, while Simos [9] and Hairer and Lubich [10] further established their theoretical and practical advantages for periodic systems. However, most symmetric multi-step methods are implicit in nature, which makes them computationally expensive, often requiring iterative predictorcorrector implementations that increase the overall cost. In contrast, non-symmetric multi-step methods can provide a more computationally efficient alternative, particularly when the phase and amplification errors are carefully analyzed and minimized during their derivation. Such methods can achieve accuracy comparable or even better than symmetric schemes for oscillatory solutions, while remaining explicit, avoiding costly correction steps, and employing in more suitable parallel computing setup, block methods, thereby offering a practical balance between efficiency and precision in long-time integrations. Recently, the family of block methods, including phase-fitted and amplification-fitted methods, has been developed for first order ODEs in [11, 12].

2 Family of Explicit Block Methods

In this section, a family of explicit block methods is developed for the direct solution of second-order ordinary differential equations (ODEs). Consider the following second-order ODE of the form

$$w'' = g(x, w), \tag{2}$$

with initial conditions $w(x_0) = w_0$ and $w'(x_0) = w_1$. The explicit block formulation with s-points approximates the solution vector $W_i = [w_{si}, w_{si+1}, \ldots, w_{si+s-1}]^T$ using linear combinations of previous solution values and derivative evaluations as follows

$$W_{i+1} = C_1 W_i + h^2 C_2 G_i, \tag{3}$$

where G_i is column matrix of double derivative evaluation. By selecting suitable coefficients, the local truncation error and phase properties can be optimized for oscillatory problems. For the method to be zero-stable, the characteristic polynomial must have roots within or on the unit circle in the complex plane and simple unit-modulus roots, this condition will leads to the following choice of coefficient matrix:

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 0 & -5 & 6 \\ 0 & 0 & 0 & 0 & -6 & 7 \end{bmatrix}. \tag{4}$$

A Taylor series expansion of $W_{n+1} - C_1 W_n - h^2 C_2 G_n$, followed by equating the coefficients of $w'(x), w''(x), \ldots, w^{(vii)}(x)$ to zero in each row, produces a system of algebraic equations. Solving this system symbolically in Mathematica software provides the coefficient matrix shown below:

$$C_2 = \begin{bmatrix} -\frac{3}{40} & \frac{109}{240} & -\frac{23}{20} & \frac{187}{120} & -\frac{133}{120} & \frac{317}{240} \\ -\frac{353}{240} & \frac{1051}{120} & -\frac{2599}{120} & \frac{1703}{60} & -\frac{4913}{240} & \frac{227}{24} \\ -\frac{581}{60} & \frac{6797}{120} & -\frac{4097}{30} & \frac{10331}{60} & -\frac{7009}{60} & \frac{4829}{120} \\ -\frac{973}{24} & \frac{2803}{12} & -\frac{6619}{12} & \frac{1349}{2} & -\frac{3487}{8} & \frac{1559}{12} \\ -\frac{3127}{24} & \frac{35579}{48} & -\frac{20651}{12} & \frac{49309}{24} & -\frac{30715}{24} & \frac{16811}{48} \\ -\frac{84163}{240} & \frac{236857}{120} & -\frac{542293}{120} & \frac{317581}{60} & -\frac{769331}{240} & \frac{99541}{120} \end{bmatrix}. \tag{5}$$

The standard block method ($M^{(1)}$) with $C_2^{(1)}$ given in (5) is written as follows

$$W_{n+1} = C_1 W_n + (h^2) C_2^{(1)} G_n, \tag{6}$$

and has local truncation error:

$$LTE = \begin{bmatrix} \frac{863h^8 w^{(8)}(x)}{12096} + O\left(h^9\right) \\ \frac{30943w^{(8)}(x)h^8}{20160} + O\left(h^9\right) \\ \frac{112969w^{(8)}(x)h^8}{10080} + O\left(h^9\right) \\ \frac{312889w^{(8)}(x)h^8}{6048} + O\left(h^9\right) \\ \frac{733855w^{(8)}(x)h^8}{4032} + O\left(h^9\right) \\ \frac{306815}{576} h^8 w^{(8)}(x) + O\left(h^9\right) \end{bmatrix}.$$

Phase-lag and Amplification error Definition
Given that the theoretical solution of the scalar test equation

$$w'' = -\mu^2 w \tag{7}$$

at $t = h$ is $\exp(\pm i\mu h)$, or equivalently $\exp(\pm i\delta)$, and that the numerical solution of the corresponding test equation (7) at $t = h$ is $\exp(\pm i\varphi(\delta))$, the *phase lag* is defined as

$$\psi = \delta - \varphi(\delta).$$

If the phase lag satisfies

$$\psi = \mathcal{O}(\delta^{\,q+1}) \quad \text{as} \quad \delta \to 0,$$

then the *order of the phase lag* is said to be q.
Theorem(Direct formulas for Amplification Error and Phase Lag)
Let $\delta = \mu h$ and define the column vectors

$$\mathbf{s}_{0:5} = \begin{bmatrix} 0 \\ \sin\delta \\ \sin 2\delta \\ \sin 3\delta \\ \sin 4\delta \\ \sin 5\delta \end{bmatrix}, \quad \mathbf{s}_{6:11} = \begin{bmatrix} \sin 6\delta \\ \sin 7\delta \\ \sin 8\delta \\ \sin 9\delta \\ \sin 10\delta \\ \sin 11\delta \end{bmatrix},$$

$$\mathbf{c}_{0:5} = \begin{bmatrix} 1 \\ \cos\delta \\ \cos 2\delta \\ \cos 3\delta \\ \cos 4\delta \\ \sin 5\delta \end{bmatrix}, \quad \mathbf{c}_{6:11} = \begin{bmatrix} \cos 6\delta \\ \cos 7\delta \\ \cos 8\delta \\ \cos 9\delta \\ \sin 10\delta \\ \sin 11\delta \end{bmatrix}.$$

Introduce the diagonal matrices

$$\varDelta = \mathrm{diag}(6, 7, 8, 9, 10, 11), \quad \varGamma = \mathrm{diag}(0, 1, 2, 3, 4, 5),$$

and matrices

$$F = -\varDelta + (C_1 - \delta^2 C_2)\,\varGamma, \qquad E = \varDelta^2 - (C_1 - \delta^2 C_2)\,\varGamma^2.$$

Then the amplification factor and the phase lag admit the compact formulas

$$AF = F^{-1}\big(\mathbf{s}_{6:11} - (C_1 - \delta^2 C_2)\mathbf{s}_{0:5}\big), \tag{8}$$

$$PhEr = E^{-1}\big(\mathbf{c}_{6:11} - (C_1 - \delta^2 C_2)\mathbf{c}_{0:5}\big). \tag{9}$$

Proof Applying the method (3) to the test equation (7) yields

$$\begin{bmatrix} w_{n+6} \\ w_{n+7} \\ w_{n+8} \\ w_{n+9} \\ w_{n+10} \\ w_{n+11} \end{bmatrix} = C_1 \begin{bmatrix} w_n \\ w_{n+1} \\ w_{n+2} \\ w_{n+3} \\ w_{n+4} \\ w_{n+5} \end{bmatrix} - (\mu h)^2 C_2 \begin{bmatrix} w_n \\ w_{n+1} \\ w_{n+2} \\ w_{n+3} \\ w_{n+4} \\ w_{n+5} \end{bmatrix}. \tag{10}$$

Its characteristic equation is therefore

$$\begin{bmatrix} \lambda^6 \\ \lambda^7 \\ \lambda^8 \\ \lambda^9 \\ \lambda^{10} \\ \lambda^{11} \end{bmatrix} = (C_1 - \delta^2 C_2) \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \end{bmatrix}, \qquad \delta = \mu h. \tag{11}$$

Using the oscillatory ansatz

$$\lambda^n = e^{\pm i n\varphi(\delta)} = \cos(n\varphi(\delta)) \pm i\sin(n\varphi(\delta)),$$

equation (11) becomes

$$\begin{bmatrix} \cos(5\varphi(\delta)) \pm i\sin(5\varphi(\delta)) \\ \cos(6\varphi(\delta)) \pm i\sin(6\varphi(\delta)) \\ \cos(7\varphi(\delta)) \pm i\sin(7\varphi(\delta)) \\ \cos(8\varphi(\delta)) \pm i\sin(8\varphi(\delta)) \\ \cos(9\varphi(\delta)) \pm i\sin(9\varphi(\delta)) \end{bmatrix} = (C_1 - \delta^2 C_2) \begin{bmatrix} 1 \\ \cos(\varphi(\delta)) \pm i\sin(\varphi(\delta)) \\ \cos(2\varphi(\delta)) \pm i\sin(2\varphi(\delta)) \\ \cos(3\varphi(\delta)) \pm i\sin(3\varphi(\delta)) \\ \cos(4\varphi(\delta)) \pm i\sin(4\varphi(\delta)) \end{bmatrix}. \tag{12}$$

Substituting the expansions [4],

$$\cos(k\varphi(\delta)) = \cos(k\delta) + k^2 d_k\, \delta^{q+2} + O(\delta^{q+4}), \qquad \sin(k\varphi(\delta)) = \sin(k\delta) - k e_k\, \delta^{q+1} + O(\delta^{q+3}),$$

equation (12) becomes

$$\mathbf{c}_{6:11} + \delta^{q+2}\Delta^2\mathbf{d}_{6:11} \ \pm i\big(\mathbf{s}_{6:11} - \delta^{q+1}\Delta\mathbf{e}_{6:11}\big) = (C_1 - \delta^2 C_2)\Big(\mathbf{c}_{0:5} + \delta^{q+2}\Gamma^2\mathbf{d}_{0:5} \pm i(\mathbf{s}_{0:5} - \delta^{q+1}\Gamma\mathbf{e}_{0:5})\Big).$$

Separating imaginary parts gives

$$\mathbf{s}_{6:11} - \delta^{q+1}\Delta\mathbf{e}_{6:11} = (C_1 - \delta^2 C_2)(\mathbf{s}_{0:5} - \delta^{q+1}\Gamma\mathbf{e}_{0:5}).$$

Collecting coefficient of δ^{q+1}, we obtain

$$-\delta^{q+1}(\Delta - \Gamma(C_1 - \delta^2 C_2))\mathbf{e} = -\mathbf{s}_{6:11} + (C_1 - \delta^2 C_2)\mathbf{s}_{0:5}.$$

Solving for $-\delta^{q+1}\mathbf{e}$ gives the amplification-factor formula

$$AF = F^{-1}\big(\mathbf{s}_{6:11} - (C_1 - \delta^2 C_2)\mathbf{s}_{0:5}\big)\,, \qquad F = -\Delta + (C_1 - \delta^2 C_2)\Gamma.$$

Likewise, extracting the real part yields

$$-\delta^{q+2}\big(\Delta^2 - (C_1 - \delta^2 C_2)\Gamma^2\big)\mathbf{d} = -\mathbf{c}_{6:11} + (C_1 - \delta^2 C_2)\mathbf{c}_{0:5},$$

which gives the phase-lag relation

$$-\delta^{q+2}\mathbf{d} = E^{-1}\big(\mathbf{c}_{6:11} - (C_1 - \delta^2 C_2)\mathbf{c}_{0:5}\big)\,,$$

with

$$E = \Delta^2 - (C_1 - \delta^2 C_2)\Gamma^2.$$

after undergoing Taylor's expansion, amplification error and phase error for method $M^{(1)}$ is

$$AF = \begin{bmatrix} -\frac{2551\delta^7}{60480} + O\left(\delta^8\right) \\ -\frac{35279\delta^7}{120960} + O\left(\delta^8\right) \\ -\frac{1064723\delta^7}{1028160} + O\left(\delta^8\right) \\ -\frac{211661\delta^7}{75600} + O\left(\delta^8\right) \\ -\frac{36924473\delta^7}{5745600} + O\left(\delta^8\right) \\ -\frac{1992043\delta^7}{151200} + O\left(\delta^8\right) \end{bmatrix}, \; PhEr = \begin{bmatrix} \frac{863\delta^8}{24192} + O\left(\delta^9\right) \\ \frac{30943\delta^8}{120960} + O\left(\delta^9\right) \\ \frac{112969\delta^8}{120960} + O\left(\delta^9\right) \\ \frac{312889\delta^8}{120960} + O\left(\delta^9\right) \\ \frac{146771\delta^8}{24192} + O\left(\delta^9\right) \\ \frac{306815\delta^8}{24192} + O\left(\delta^9\right) \end{bmatrix}.$$

Thus, the block method $M^{(1)}$ corresponding to (5) is of the sixth-order phase lag and sixth-order amplification factor.

2.1 Minimal phase error, amplification fitted method

By employing the coefficient matrix C_1 from (4) together with the direct expression (8), the steps below outline the construction of an amplification-fitted block method with reduced phase-lag.

Algorithm 1 Procedure for an Amplification-Fitted Block Method

1: Suppress the amplification factor to zero and derive the corresponding phase-lag expression.
2: Perform a Taylor expansion of the phase-lag and minimize its dominant terms.
3: Determine the remaining coefficients to further decrease the local truncation error.

Following the algorithm, the resulting expressions obtained by setting the amplification error to zero are given in http://theodoresimos.org/. Furthermore, minimizing the phase error by enforcing the coefficients of δ^2, δ^4, and δ^6 to vanish yields the updated element values listed in the table. Finally, using the local truncation error in the last step of the algorithm leads to the following coefficient matrices.

Table 1: Relations obtained from eliminating phase terms

δ^2	
$c_5 = -5c_1 - 4c_2 - 3c_3 - 2c_4$	$c_{10} = -\frac{c_{11}}{2} - \frac{5c_7}{2} - 2c_8 - \frac{3c_9}{2} - \frac{1}{2}$
$c_{17} = -5c_{13} - 4c_{14} - 3c_{15} - 2c_{16} - 4$	$c_{23} = -5c_{19} - 4c_{20} - 3c_{21} - 2c_{22} - 10$
$c_{29} = -5c_{25} - 4c_{26} - 3c_{27} - 2c_{28} - 20$	$c_{35} = -5c_{31} - 4c_{32} - 3c_{33} - 2c_{34} - 35$
δ^4	
$c_4 = -20c_1 - 10c_2 - 4c_3$	$c_{11} = 35c_7 + 16c_8 + 5c_9 - \frac{5}{6}$
$c_{16} = -20c_{13} - 10c_{14} - 4c_{15} - \frac{4}{3}$	$c_{22} = -20c_{19} - 10c_{20} - 4c_{21} - \frac{41}{6}$
$c_{28} = -20c_{25} - 10c_{26} - 4c_{27} - \frac{68}{3}$	$c_{34} = -20c_{31} - 10c_{32} - 4c_{33} - \frac{707}{12}$
δ^6	
$c_3 = -21c_1 - 6c_2$	$c_9 = -21c_7 - 6c_8 + \frac{1}{240}$
$c_{15} = -21c_{13} - 6c_{14} - \frac{1}{15}$	$c_{21} = -21c_{19} - 6c_{20} - \frac{35}{24}$
$c_{27} = -21c_{25} - 6c_{26} - \frac{29}{3}$	$c_{33} = -21c_{31} - 6c_{32} - \frac{1945}{48}$

By optimizing the local truncation error, the following coefficient matrix $C_2 = C_2^{(2)}$ is obtained

$$C_2^{(2)} = \begin{bmatrix} -\frac{3}{40} & \frac{109}{240} & -\frac{23}{20} & \frac{187}{120} & -\frac{133}{120} & c_{16} \\ -\frac{353}{240} & \frac{1051}{120} & -\frac{2599}{120} & \frac{1703}{60} & -\frac{4913}{240} & c_{26} \\ -\frac{581}{60} & \frac{6797}{120} & -\frac{4097}{30} & \frac{10331}{60} & -\frac{7009}{60} & c_{36} \\ -\frac{973}{24} & \frac{2803}{12} & -\frac{6619}{12} & \frac{1349}{2} & -\frac{3487}{8} & c_{46} \\ -\frac{3127}{24} & \frac{35579}{48} & -\frac{20651}{12} & \frac{49309}{24} & -\frac{30715}{24} & c_{56} \\ -\frac{84163}{240} & \frac{236857}{120} & -\frac{542293}{120} & \frac{317581}{60} & -\frac{769331}{240} & c_{66} \end{bmatrix}$$

where remaining elements are given in http://theodoresimos.org/.

The phase lag for method $M^{(2)}$ is

$$PhEr = \begin{bmatrix} \frac{7\delta^8}{480} + O\left(\delta^9\right) \\ \frac{1819133\delta^8}{1814400} + O\left(\delta^9\right) \\ \frac{157453\delta^8}{453600} + O\left(\delta^9\right) \\ \frac{2742361\delta^8}{3024000} + O\left(\delta^9\right) \\ \frac{4528663\delta^8}{2268000} + O\left(\delta^9\right) \\ \frac{1010683\delta^8}{259200} + O\left(\delta^9\right) \end{bmatrix}.$$

The order of the method is 6, obtained from local truncation error,

$$LTE = \begin{bmatrix} \left(\frac{2551w''(x)\mu^6}{60480} + \frac{863w^{(8)}(x)}{12096}\right) h^8 + O\left(h^9\right) \\ \left(\frac{35279w''(x)\mu^6}{37800} + \frac{30943w^{(8)}(x)}{20160}\right) h^8 + O\left(h^9\right) \\ \left(\frac{1064723w''(x)\mu^6}{151200} + \frac{112969w^{(8)}(x)}{10080}\right) h^8 + O\left(h^9\right) \\ \left(\frac{211661w''(x)\mu^6}{6300} + \frac{312889w^{(8)}(x)}{6048}\right) h^8 + O\left(h^9\right) \\ \left(\frac{36924473w''(x)\mu^6}{302400} + \frac{733855w^{(8)}(x)}{4032}\right) h^8 + O\left(h^9\right) \\ \left(\frac{1992043w''(x)\mu^6}{5400} + \frac{306815}{576}w^{(8)}(x)\right) h^8 + O\left(h^9\right) \end{bmatrix}.$$

2.2 Method 3: Amplification-fitted and phase fitted block method

The amplification-fitted and phase-fitted block method is obtained through the following sequence of operations:

Algorithm 2 Procedure for the Phase-fitted Block Method

1: Eliminate amplification factor and the phase error.
2: Using these updated relations, recompute the local truncation error.
3: Assign values to the remaining undetermined coefficients by enforcing the minimization of this truncation error.

A full derivation based on this procedure is provided in http://theodoresimos.org/ . The coefficient matrix C_2 appearing in equation (3) for the resulting method is denoted by $C_2(3)$ and is shown below:

$$C_2^{(3)} = \begin{bmatrix} c_{11} & \frac{109}{240} & -\frac{23}{20} & \frac{187}{120} & -\frac{133}{120} & c_{16} \\ c_{21} & \frac{1051}{120} & -\frac{2599}{120} & \frac{1703}{60} & -\frac{4913}{240} & c_{26} \\ c_{31} & \frac{6797}{120} & -\frac{4097}{30} & \frac{10331}{60} & -\frac{7009}{60} & c_{36} \\ c_{41} & \frac{2803}{12} & -\frac{6619}{12} & \frac{1349}{2} & -\frac{3487}{8} & c_{46} \\ c_{51} & \frac{35579}{48} & -\frac{20651}{12} & \frac{49309}{24} & -\frac{30715}{24} & c_{56} \\ c_{61} & \frac{236857}{120} & -\frac{542293}{120} & \frac{317581}{60} & -\frac{769331}{240} & c_{66} \end{bmatrix}$$

The remaining entries of the matrix $C_2^{(3)}$ are listed in http://theodoresimos.org/ . The method is designated as $M^{(3)}$ and is of order five, since the corresponding local truncation error of the block method $M^{(3)}$ is given by

$$LTE = \begin{bmatrix} \frac{863\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8}{12096} + O\left(h^9\right) \\ \frac{30943\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8}{20160} + O\left(h^9\right) \\ \frac{112969\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8}{10080} + O\left(h^9\right) \\ \frac{312889\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8}{6048} + O\left(h^9\right) \\ \frac{733855\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8}{4032} + O\left(h^9\right) \\ \frac{306815}{576}\left(w''(x)\mu^6+w^{(8)}(x)\right)h^8 + O\left(h^9\right) \end{bmatrix}.$$

2.3 Method 4: Amplification-fitted and phase fitted block method with vanished first derivative of phase-error

Motivated by the fact that a method with zero phase-lag may still suffer from instability causing error amplification under small deviations if the first derivative of the phase error does not vanish, the following procedure constructs a method in which both the phase error and its first derivative are forced to be zero.

Algorithm 3 Algorithm for the Phase-Lag and Derivative-Fitted Method

1: Compute the amplification factor (AF) together with the phase error ($PhEr$).
2: Evaluate the first derivative of the phase error with respect to the frequency parameter δ.
3: Formulate the conditions under which $AF = 0$, $PhEr = 0$, and $PhEr' = 0$, and derive the corresponding coefficient relations.
4: Finalize the remaining free coefficients by minimizing the local truncation error (LTE).

The algorithm results into following coefficient matrix for method $M^{(4)}$:

$$C_2^{(4)} = \begin{bmatrix} c_{11} & \frac{109}{240} & -\frac{23}{20} & \frac{187}{120} & c_{15} & c_{16} \\ c_{21} & \frac{1051}{120} & -\frac{2599}{120} & c_{24} & -\frac{4913}{240} & c_{26} \\ c_{31} & \frac{6797}{120} & -\frac{4097}{30} & \frac{10331}{60} & c_{35} & c_{36} \\ c_{41} & \frac{2803}{12} & -\frac{6619}{12} & \frac{1349}{2} & c_{45} & c_{46} \\ c_{51} & \frac{35579}{48} & -\frac{20651}{12} & \frac{49309}{24} & c_{55} & c_{56} \\ c_{61} & \frac{236857}{120} & -\frac{542293}{120} & \frac{317581}{60} & c_{65} & c_{66} \end{bmatrix},$$

with detailed expressions for elements given in http://theodoresimos.org/ and the local truncation error given as

$$LTE = \begin{bmatrix} \frac{\left(-1783w''(x)\mu^6 - 2646w^{(4)}(x)\mu^4 + 863w^{(8)}(x)\right)h^8}{12096} + O\left(h^9\right) \\ \frac{\left(-329138w''(x)\mu^6 - 545739w^{(4)}(x)\mu^4 + 216601w^{(8)}(x)\right)h^8}{141120} + O\left(h^9\right) \\ \frac{\left(-201937w''(x)\mu^6 - 314906w^{(4)}(x)\mu^4 + 112969w^{(8)}(x)\right)h^8}{10080} + O\left(h^9\right) \\ \frac{\left(-5098193w''(x)\mu^6 - 8227083w^{(4)}(x)\mu^4 + 3128890w^{(8)}(x)\right)h^8}{60480} + O\left(h^9\right) \\ \frac{\left(-5388051w''(x)\mu^6 - 9057326w^{(4)}(x)\mu^4 + 3669275w^{(8)}(x)\right)h^8}{20160} + O\left(h^9\right) \\ \frac{\left(-4006631w''(x)\mu^6 - 7074781w^{(4)}(x)\mu^4 + 3068150w^{(8)}(x)\right)h^8}{5760} + O\left(h^9\right) \end{bmatrix},$$

2.4 Method 5: Amplification-fitted and phase fitted block method with vanished first derivative of amplification-factor

The development of the method proceeds through the steps below:

1: Compute the amplification factor (AF) and the phase error ($PhEr$).
2: Evaluate the first derivative of the amplification error with respect to δ.
3: Establish the conditions under which $AF = 0$, $PhEr = 0$, and $AF' = 0$, and derive the resulting coefficient relations.
4: Obtain the remaining free coefficients by minimizing the local truncation error LTE.

The above procedure leads to following matrix

$$C_2^{(5)} = \begin{bmatrix} c_{11} & \frac{109}{240} & -\frac{23}{20} & \frac{187}{120} & c_{15} & c_{16} \\ c_{21} & \frac{1051}{120} & -\frac{2599}{120} & \frac{1703}{60} & c_{25} & c_{26} \\ c_{31} & \frac{6797}{120} & -\frac{4097}{30} & \frac{10331}{60} & c_{35} & c_{36} \\ c_{41} & \frac{2803}{12} & -\frac{6619}{12} & \frac{1349}{2} & c_{45} & c_{46} \\ c_{51} & \frac{35579}{48} & -\frac{20651}{12} & \frac{49309}{24} & c_{55} & c_{56} \\ c_{61} & \frac{236857}{120} & -\frac{542293}{120} & \frac{317581}{60} & c_{65} & c_{66} \end{bmatrix}.$$

http://theodoresimos.org/ provides the expressions for the remaining elements. This method $M^{(5)}$ is of order 6 as corresponding LTE is given as follows:

$$LTE = \begin{bmatrix} \frac{\left(-1688w''(x)\mu^6-2551w^{(4)}(x)\mu^4+863w^{(8)}(x)\right)h^8}{12096} + O\left(h^9\right) \\ \frac{\left(-189403w''(x)\mu^6-282232w^{(4)}(x)\mu^4+92829w^{(8)}(x)\right)h^8}{60480} + O\left(h^9\right) \\ \frac{\left(-725816w''(x)\mu^6-1064723w^{(4)}(x)\mu^4+338907w^{(8)}(x)\right)h^8}{30240} + O\left(h^9\right) \\ \frac{\left(-3515419w''(x)\mu^6-5079864w^{(4)}(x)\mu^4+1564445w^{(8)}(x)\right)h^8}{30240} + O\left(h^9\right) \\ \frac{\left(-25916648w''(x)\mu^6-36924473w^{(4)}(x)\mu^4+11007825w^{(8)}(x)\right)h^8}{60480} + O\left(h^9\right) \\ \frac{\left(-11334119w''(x)\mu^6-15936344w^{(4)}(x)\mu^4+4602225w^{(8)}(x)\right)h^8}{8640} + O\left(h^9\right) \end{bmatrix},$$

2.5 Method 6: Amplification-fitted and phase fitted block method with vanished first derivative of amplification-factor and phase error

The method is derived through the following sequence of steps:

1: Compute the amplification factor (AF) and the phase error ($PhEr$).
2: Evaluate the first derivatives of AF and $PhEr$ with respect to δ.
3: Impose the conditions $AF = 0$, $PhEr = 0$, $AF' = 0$, and $PhEr' = 0$ to obtain the defining relations of the block method.
4: Determine the remaining coefficients by minimizing the local truncation error (LTE).

This procedure yields the following coefficient matrix: $C_2^{(6)}$:

$$C_2^{(6)} = \begin{bmatrix} c_{11} & \frac{109}{240} & -\frac{23}{20} & c_{14} & c_{15} & c_{16} \\ c_{21} & \frac{1051}{120} & -\frac{2599}{120} & c_{24} & c_{25} & c_{26} \\ c_{31} & \frac{6797}{120} & -\frac{4097}{30} & c_{34} & c_{35} & c_{36} \\ c_{41} & \frac{2803}{12} & -\frac{6619}{12} & c_{44} & c_{45} & c_{46} \\ c_{51} & \frac{35579}{48} & -\frac{20651}{12} & c_{54} & c_{55} & c_{56} \\ c_{61} & \frac{236857}{120} & -\frac{542293}{120} & c_{64} & c_{65} & c_{66} \end{bmatrix},$$

where the members of the matrix are given in http://theodoresimos.org/. Local Truncation error of the method $M^{(6)}$ is

$$LTE = \begin{bmatrix} -\frac{863\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8}{12096} + O\left(h^9\right) \\ -\frac{30943\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8}{20160} + O\left(h^9\right) \\ -\frac{112969\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8}{10080} + O\left(h^9\right) \\ -\frac{312889\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8}{6048} + O\left(h^9\right) \\ -\frac{733855\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8}{4032} + O\left(h^9\right) \\ -\frac{306815}{576}\left(2w''(x)\mu^6+3w^{(4)}(x)\mu^4-w^{(8)}(x)\right)h^8 + O\left(h^9\right) \end{bmatrix}.$$

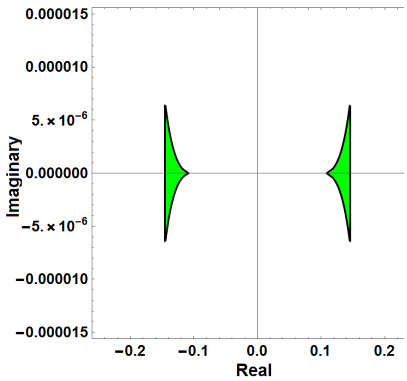

Fig. 1: Region of Stability for $M^{(1)}$

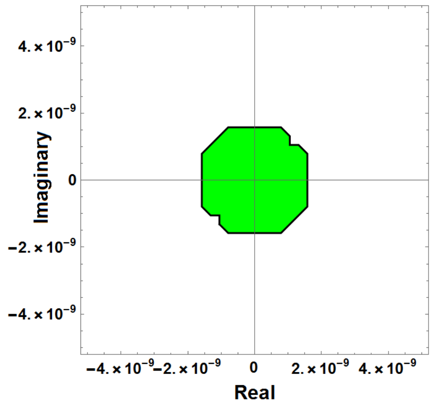

Fig. 2: Region of Stability for $M^{(2)}$ $(AF = 0)$ with $\delta = 1$

3 Stability

The stability of the block schemes is analysed using the test equation $w'' = -\mu^2 w$. When the method is applied to this problem, the numerical solution satisfies

$$U_{n+1} = (A - H^2 B)U_n, \qquad H = \mu h,$$

where the amplification matrix governs the propagation of errors between successive blocks.

Stability is determined by the eigenvalues of this matrix. These are obtained from the characteristic equation

$$Z^6 + S_1(H)Z^5 + S_2(H)Z^4 + S_3(H)Z^3 + S_4(H)Z^2 + S_5(H)Z + S_6(H) = 0,$$

whose coefficients depend on the block scheme and the parameter H. The method is stable for a given H if all roots Z satisfy $|Z| \leq 1$.

The stability regions are computed by varying H, evaluating the roots of the characteristic polynomial, and identifying the admissible range for which the stability condition holds. Symbolic and numerical computations were performed in Mathematica. Figures 1-10 display the resulting stability regions for methods $M^{(1)}$ - $M^{(6)}$ and different values of the tuning parameter δ, illustrating the influence of both step size and parameter choice on the stability of oscillatory solutions.

4 Numerical Experiments

A broad set of test problems is used to assess the performance of the proposed schemes, including oscillatory secondorder ODEs and selected PDE benchmarks. Accuracy is measured using the logarithmic error indicator

$$\text{Errham} = -\frac{\ln(|w(x_i) - w_i|)}{\ln(10)}.$$

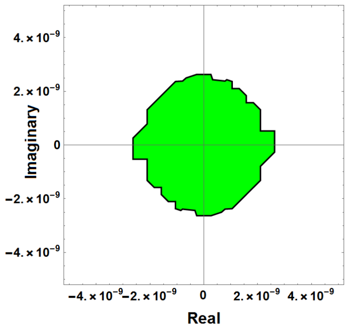

Fig. 3: Region of Stability for $M^{(2)}$ $(AF = 0)$ with $\delta = 1000$

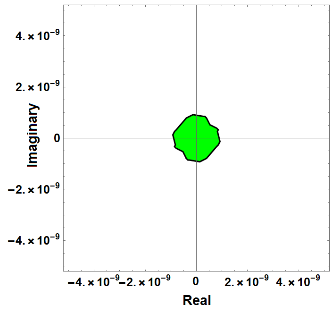

Fig. 4: Region of Stability for $M^{(3)}$ $(AF = 0, PE = 0)$ with $\delta = 1$

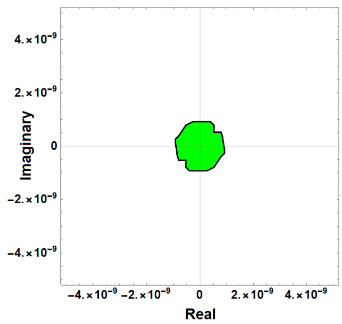

Fig. 5: Region of Stability for $M^{(3)}$ $(AF = 0, PE = 0)$ with $\delta = 1000$

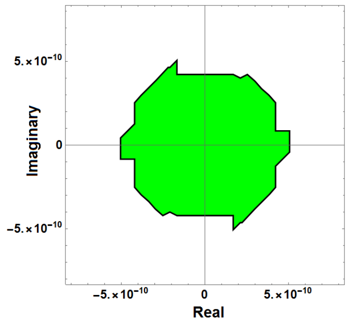

Fig. 6: Stability region for $M^{(4)}$ $(AF = 0, PE = 0, D[PE] = 0)$ with $\delta = 1$

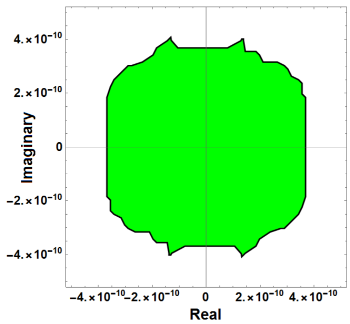

Fig. 7: Region of Stability for $M^{(4)}$ $(AF = 0, PE = 0, D[PE] = 0)$ with $\delta = 1000$

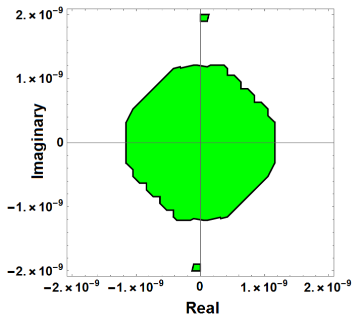

Fig. 8: Region of Stability for $M^{(5)}$ $(AF = 0, PE = 0, D[AF] = 0)$ with $\delta = 1$

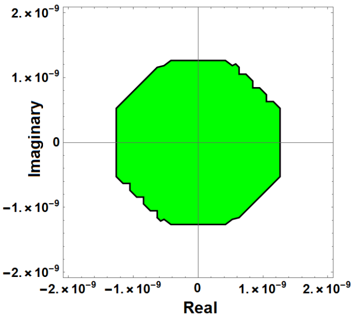

Fig. 9: Region of Stability for $M^{(5)}$ $(AF = 0, PE = 0, D[AF] = 0)$ with $\delta = 1000$

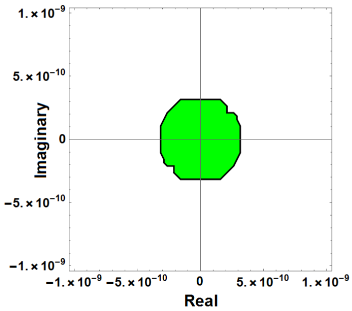

Fig. 10: Region of Stability for $M^{(6)}$ $(AF = 0, PE = 0, D[PE] = 0, D[AF] = 0)$ with $\delta = 1$

The ODE experiments are carried out in MATHEMATICA, while PDE simulations are performed in MATLAB. Comparisons are made with standard solvers, namely the embedded RungeKutta methods **RKCash** [14] and **RKFehl** [15], and the 12-stage sixth-order symplectic RungeKuttaNystrm method **12s6r** [13].

To evaluate oscillation-adapted strategies, the test set also includes trigonometrically fitted methods (**T1**, **T2**), an explicit Obrechkoff-type scheme (**T3**), and a fifth-order RungeKuttaNystrm method (**T4**) [16–20]. Since block methods require starting values, the first four points are generated using the MATLAB solver *ode45*. For frequency-dependent integrators, the choice of the dominant frequency parameter μ plays a key role; when it is not directly available from the model, established estimation procedures from the literature are employed [2, 3].

Example 1. Considering the following spring equation

$$w''(x) = -81w(x), \ x \in [0, 2] \tag{13}$$

which has exact solution $w(x) = 2\cos(9x) + (1/9)\sin(9x)$. In Table 2, the developed methods are compared only with the 12-stage sixth order explicit symplectic RungeKutta-Nystrm method (referred as 12s6r) [13], as the 9 and 10 stage variants, while occasionally comparable, do not surpass the accuracy of the 12-stage method in prior studies.

Table 2: Comparison of absolute errors for various methods at different choice of step-size at $x = 2$

Method	$3h = 2^{-4}$	$3h = 2^{-5}$	$3h = 2^{-6}$
12s6r	2.07×10^{-9}	3.34×10^{-11}	5.30×10^{-13}
$M^{(1)}$	1.58	2.92×10^{-4}	8.13×10^{-6}
$M^{(2)}$	0.3911	1.21×10^{-4}	2.71×10^{-6}
$M^{(3)}$	3.83×10^{-10}	9.11×10^{-13}	9.91×10^{-13}
$M^{(4)}$	7.63×10^{-10}	5.60×10^{-12}	2.95×10^{-13}
$M^{(5)}$	2.11×10^{-9}	1.72×10^{-10}	3.47×10^{-10}
$M^{(6)}$	6.02×10^{-10}	3.07×10^{-11}	9.60×10^{-11}

Example 2: The following second order differential equation is considered

$$w''(x) + 9\,w(x) = 3\sin(3x), \qquad x \in [0, 40\pi]$$

with the initial conditions

$$w(0) = 1, \qquad w'(0) = 3,$$

and the true solution is given below

$$w(x) = \frac{7}{6}\sin(3x) + \cos(3x) - \frac{1}{2}\,x\cos(3x). \tag{14}$$

Table 3 lists absolute errors for different values of μ and are compared with trigonometrically-fitted methods T1, T2 and T3.

Table 3: Numerical results for different values of μ

Method	$\mu = 2.95$	$\mu = 3.00$	$\mu = 3.05$
T1	3.03×10^{-4}	1.09×10^{-6}	3.38×10^{-4}
T2	4.48×10^{-3}	4.47×10^{-3}	4.46×10^{-3}
T3	3.47×10^{-6}	1.35×10^{-41}	3.64×10^{-6}
$M^{(1)}$	1.59×10^{-6}	1.59×10^{-6}	1.59×10^{-6}
$M^{(2)}$	1.13×10^{-6}	1.08×10^{-6}	1.03×10^{-6}
$M^{(3)}$	5.78×10^{-7}	4.70×10^{-7}	3.52×10^{-7}
$M^{(4)}$	1.63×10^{-7}	1.20×10^{-7}	7.93×10^{-8}
$M^{(5)}$	1.31×10^{-6}	1.01×10^{-6}	1.44×10^{-7}
$M^{(6)}$	3.52×10^{-8}	3.21×10^{-9}	2.76×10^{-8}

Example 3: A cubic nonlinear oscillator discussed in [21] is described by

$$\frac{d^2w}{dx^2} + w = \gamma\,w^3, \qquad \gamma = 10^{-3},$$

with initial conditions

$$w(0) = 1, \qquad w'(0) = 0.$$

According to [21], an analytical expansion retaining terms up to γ^3 is

$$w(x) = \cos(\mu x) + \frac{\gamma}{128}[\cos(3\mu x) + \cos(\mu x)] + \mathcal{O}(\gamma^3).$$

and the modified frequency associated with this system is

$$\mu = \sqrt{1 - \tfrac{3}{4}\gamma}\,.$$

To assess numerical performance, the error is evaluated at $x = 2000\pi$. Table 4 presents the errors produced by the proposed sixth-order block methods, along with the reference schemes T3 and T4. A fixed step size $h = \pi/27$ is used for all members of the block-method family.

Table 4: Error estimate for different methods

Method	L_2 Error
$M^{(1)}$	0.159939
$M^{(2)}$	0.0353436
$M^{(3)}$	2.65×10^{-6}
$M^{(4)}$	8.41×10^{-8}
$M^{(5)}$	3.90×10^{-6}
$M^{(6)}$	3.92×10^{-6}
T3	1.50×10^{-4}
T4	1.48×10^{-4}

Example 4: Stiefel and Bettis [22] studied the almost periodic orbit problem, and it is as follows:

$$\begin{aligned} w_1''(x) &= -w_1(x) + 0.001\cos(x), \quad w_1(0) = 1, \quad w_1'(0) = 0, \\ w_2''(x) &= -w_2(x) + 0.001\sin(x), \quad w_2(0) = 0, \quad w_2'(0) = 0.9995. \end{aligned} \tag{15}$$

The problem admits the following exact solution

$$\begin{aligned} w_1(x) &= \cos(x) + 0.0005x\sin(x), \\ w_2(x) &= \sin(x) - 0.0005x\cos(x). \end{aligned} \tag{16}$$

Table 5: Numerical results at $x = 100$ for example 4.

Methods	$\mathbf{L}_\infty$ error	CPU time	$\mathbf{L}_\infty$ error	CPU time
12s6r	7.69×10^{-10}	5.0625	1.02×10^{-10}	9.0625
$M^{(1)}$	4.68×10^{-3}	0.046875	7.80×10^{-4}	0.03125
$M^{(2)}$	1.59×10^{-3}	0.015625	3.27×10^{-4}	0.046875
$M^{(3)}$	1.44×10^{-5}	0.03125	2.41×10^{-6}	0.03125
$M^{(4)}$	5.28×10^{-6}	0.046875	5.33×10^{-7}	0.015625
$M^{(5)}$	3.54×10^{-6}	0.015625	3.39×10^{-7}	0.015625
$M^{(6)}$	8.44×10^{-11}	0.03125	1.30×10^{-11}	0.03125

Table 5 reports the absolute errors at $x = 100$, showing a clear improvement in accuracy as the methods progress from $M^{(1)}$ to $M^{(6)}$. The method `12s6r` performs worse than $M^{(6)}$, which is expected due to the non-autonomous nature of the problem. To assess long-term behavior, Figure 11 presents the performance of the proposed methods $M^{(1)}M^{(6)}$ for $\mu = 1$ up to $x = 10000$. The comparative observations are summarized below:

- The RKFehl scheme initially exhibits the poorest accuracy among the Runge–Kutta methods considered, although its performance improves gradually.
- The classical fourth-order Runge-Kutta method (RK4) achieves better accuracy than both RKFehl and RKCash but shows no further improvement beyond its initial performance level.
- The block method $M^{(1)}$ outperforms all tested Runge-Kutta schemes, highlighting the advantages of the block formulation.
- Method $M^{(2)}$ provides a modest improvement over $M^{(1)}$.

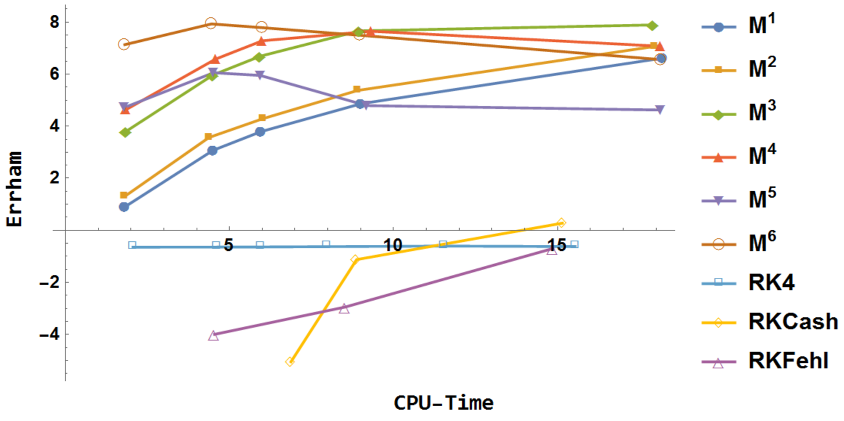

Fig. 11: Comparison of CPU-time and accuracy for example 4

- Method $M^{(6)}$ attains the highest initial accuracy, while method $M^{(3)}$ tend to surpasses it as the computational effort increases.
- Methods $M^{(4)}$ and $M^{(3)}$ deliver comparable accuracy, both notably outperforming $M^{(1)}$.
- Method $M^{(5)}$ performs slightly worse than $M^{(4)}$.
- Overall, $M^{(6)}$ emerges as the most accurate and efficient method in the family, offering an excellent balance between accuracy and computational cost.

Example 5. We consider a modified variant of the classical Franco–Palacios problem [23], governed by the coupled system

$$w''(x) = -w(x) + \varepsilon\cos(\vartheta x), \qquad w(0) = 1, \quad w'(0) = 0, \tag{17}$$

$$v''(x) = -v(x) + \varepsilon\sin(\vartheta x), \qquad v(0) = 0, \quad v'(0) = 1. \tag{18}$$

A closed-form representation of the solution to Eqs. (17)–(18) can be expressed as

$$w(x) = \frac{1-\varepsilon-\vartheta^2}{1-\vartheta^2}\cos(x) + \frac{\varepsilon}{1-\vartheta^2}\cos(\vartheta x), \tag{19}$$

$$v(x) = \frac{1-\varepsilon\vartheta-\vartheta^2}{1-\vartheta^2}\sin(x) + \frac{\varepsilon}{1-\vartheta^2}\sin(\vartheta x). \tag{20}$$

In the numerical experiments, the parameters are set to $\varepsilon = 0.001$ and $\vartheta = 0.01$, and the integration is performed over the interval $x \in [0, 10000]$.

The main findings based on the computational results presented in Figure 12 can be summarized as follows:

- Within the classical Runge–Kutta family, the Cash–Karp scheme (CashRK) attains the highest accuracy, whereas the Fehlberg (FehlRK) and classical fourth-order RK4 variants show comparatively poorer performance, with RK4 being the least accurate.

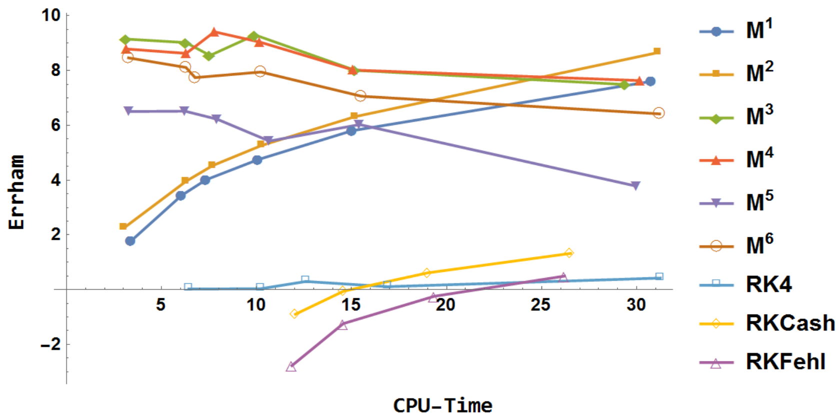

Fig. 12: CPU time versus accuracy for Example 5.

- The block method $M^{(1)}$ yields a noticeable increase in precision relative to all the Runge–Kutta methods, demonstrating the advantage of the block-based formulation.
- Method $M^{(2)}$ provides an additional improvement over $M^{(1)}$, indicating progressive refinement within the constructed family of block methods.
- The methods $M^{(3)}$, $M^{(4)}$, and $M^{(5)}$ exhibit almost indistinguishable accuracy and significantly outperform $M^{(1)}$, confirming their stable and consistent behavior.
- Although $M^{(6)}$ is more accurate than $M^{(1)}$, its performance remains slightly inferior to that of $M^{(3)}$, $M^{(4)}$, and $M^{(5)}$.

Example 6: We examine a nonlinear variant of the Sine-Gordon model governed by

$$w_{xx}(x, y) = w_{yy}(x, y) - \frac{\pi^2}{4} w(x, y) - w^2(x, y),$$

together with the initial data

$$w(0, y) = 0, \qquad \partial_x w(0, y) = \frac{\pi y}{2}, \qquad 0 \leq y \leq 1.$$

This problem possesses an exact closed-form solution, which can be written as

$$w(x, y) = y \sin\left(\frac{\pi x}{2}\right).$$

Following [24], the PDE is spatially discretized using a uniform algebraic trigonometric tension B-spline–based DQM, yielding an ODE system that is integrated with the proposed block methods. Table 6 presents a comparison with the HBBM method [25] for $x = 0.01$ and $\mu = \pi/2$, demonstrating the superior performance of the proposed methods.

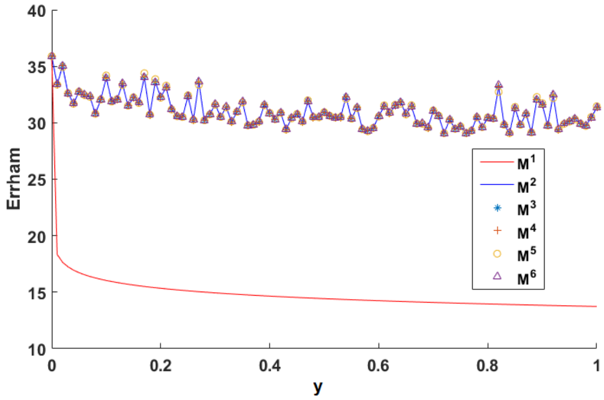

Fig. 13: Accuracy at different points of domain for example 6

Table 6: Numerical results for example 6

y	HBBM [25]	M^1	M^2	M^3	M^4	M^5	M^6
0.01	1.29×10^{-8}	1.0864×10^{-21}	1.0864×10^{-21}	1.0864×10^{-21}	1.0864×10^{-21}	1.0864×10^{-21}	1.0864×10^{-21}
0.02	2.58×10^{-8}	0	2.7105×10^{-20}	2.7105×10^{-20}	2.7105×10^{-20}	5.0882×10^{-16}	5.4210×10^{-20}
0.03	3.88×10^{-8}	0	5.4210×10^{-20}	5.4210×10^{-20}	5.4210×10^{-20}	1.0176×10^{-15}	1.6263×10^{-19}
0.04	5.17×10^{-8}	1.0842×10^{-19}	0	0	0	1.5263×10^{-15}	3.2526×10^{-19}
0.05	6.46×10^{-8}	0	1.0842×10^{-19}	1.0842×10^{-19}	1.0842×10^{-19}	2.0353×10^{-15}	3.2526×10^{-19}
0.06	7.75×10^{-8}	1.0842×10^{-19}	1.0842×10^{-19}	1.0842×10^{-19}	1.0842×10^{-19}	2.5440×10^{-15}	3.2526×10^{-19}
0.07	9.05×10^{-8}	0	2.1684×10^{-19}	2.1684×10^{-19}	2.1684×10^{-19}	3.0529×10^{-15}	4.3368×10^{-19}
0.08	1.03×10^{-7}	0	2.1684×10^{-19}	2.1684×10^{-19}	2.1684×10^{-19}	3.5616×10^{-15}	4.3368×10^{-19}
0.09	1.16×10^{-7}	0	2.1684×10^{-19}	2.1684×10^{-19}	2.1684×10^{-19}	4.0705×10^{-15}	6.5052×10^{-19}
0.10	1.29×10^{-7}	2.1684×10^{-19}	2.1684×10^{-19}	2.1684×10^{-19}	2.1684×10^{-19}	4.5790×10^{-15}	8.6736×10^{-19}

Example 7: A nonlinear variant of the Sine–Gordon equation is considered,

$$w_{xx}(x,y) - w_{yy}(x,y) + \sin\big(w(x,y)\big) = 0, \qquad (x,y) \in [0,1],$$

with initial and boundary data generated consistently from a known closed-form solution.

The analytical solution associated with this model is

$$w(x,y) = 4\tan^{-1}\big(x\,\mathrm{sech}(y)\big).$$

Table 7 provides a numerical comparison between the proposed block methods and the HBBM scheme [25], while Figure 14 presents the corresponding accuracy curves. The results demonstrate a clear improvement in precision for the block family, with $M^{(6)}$ yielding the smallest error across the tested configurations and $M^{(5)}$ as least accurate among the derived methods.

5 Conclusion

The proposed block methods show a clear improvement over classical Runge–Kutta, trigonometrically fitted, and symplectic schemes. The initial member

y	HBBM [25]	$M^{(1)}$	$M^{(2)}$	$M^{(3)}$	$M^{(4)}$	$M^{(5)}$	$M^{(6)}$
0.1	2.61×10^{-11}	1.25×10^{-16}	1.32×10^{-16}	1.32×10^{-16}	1.32×10^{-16}	6.64×10^{-13}	1.53×10^{-16}
0.2	2.15×10^{-11}	5.55×10^{-17}	5.55×10^{-17}	5.55×10^{-17}	6.25×10^{-17}	6.35×10^{-13}	8.33×10^{-17}
0.3	1.49×10^{-11}	1.39×10^{-17}	1.39×10^{-17}	1.39×10^{-17}	2.08×10^{-17}	5.90×10^{-13}	3.47×10^{-17}
0.4	7.49×10^{-12}	2.78×10^{-17}	2.78×10^{-17}	2.78×10^{-17}	2.08×10^{-17}	5.34×10^{-13}	6.94×10^{-18}
0.5	5.70×10^{-13}	4.86×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	4.16×10^{-17}	4.70×10^{-13}	2.78×10^{-17}
0.6	5.06×10^{-12}	5.55×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	4.05×10^{-13}	3.47×10^{-17}
0.7	8.96×10^{-12}	4.86×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	4.16×10^{-17}	3.41×10^{-13}	3.47×10^{-17}
0.8	1.11×10^{-11}	4.86×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	4.86×10^{-17}	2.82×10^{-13}	3.82×10^{-17}
0.9	1.18×10^{-11}	3.09×10^{-16}	3.09×10^{-16}	3.09×10^{-16}	3.05×10^{-16}	2.29×10^{-13}	2.98×10^{-16}
1.0	1.14×10^{-11}	7.16×10^{-09}	7.16×10^{-09}	7.16×10^{-09}	7.16×10^{-09}	7.16×10^{-09}	7.16×10^{-09}

Table 7: Error comparison corresponding to $M^{(1)}$–$M^{(6)}$ values at different y values

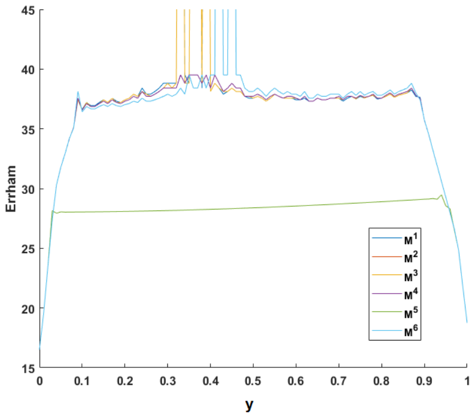

Fig. 14: Performance of the family of block methods for example 7

$M^{(1)}$ already surpasses RK4, Fehlberg, and Cash–Karp, while accuracy and efficiency steadily increase from $M^{(2)}$ to $M^{(6)}$. Among the higher variants, $M^{(3)}$, $M^{(4)}$, and $M^{(6)}$ perform better comparably to established trigonometrically fitted methods, demonstrating strong capability for oscillatory problems. Applications to partial differential equations further verify the robustness of the phase- and amplification-fitted construction. Although no single method dominates across all tests, the findings highlight the critical role of reducing phase and amplification errors in achieving reliable and efficient integration.

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