High–Order Adams–Bashforth Methods for First–Order IVPs with Oscillating Solutions

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Abstract. The theory for computing the phase–lag and amplification–factor for both explicit and implicit multistep approaches for first-order differential equations was recently established by one of the authors. The goal of this work is to design high order Adams–Bashforth algorithms that eliminate the phase–lag, amplification–factor, and their respective derivatives. The stability zones of the recently developed approaches will also be presented. Additionally, we'll discuss our findings from numerical experiments using the recently created techniques.

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1. Introduction

Numerous disciplines, such as physics, chemistry, electronics, nanotechnology, astronomy, and materials science, use equations or sets of equations that look like this to address problems:

$$\boldsymbol{\Upsilon}'(\mathbf{t}) = \mathbf{M}(\mathbf{t}, \boldsymbol{\Upsilon}),\ \boldsymbol{\Upsilon}(\mathbf{t_0}) = \boldsymbol{\Upsilon}_\mathbf{0}. \tag{1.1}$$

See also [1] and [2] for more information about the category of equations that have periodic or oscillatory solutions.

Over the last twenty years, numerous attempts have been made to find the numerical solution to the given problem or set of equations (for examples, see [3]–[7], and the related works). For a more detailed examination of the methods employed to resolve (1.1) with solutions exhibiting oscillating behavior, see [8], [3], [4], and the referenced works; for instance, Quinlan and Tremaine, [5], and [9]. The numerical methods that have been published to solve (1.1) share some characteristics, the most noticeable of which is that they are hybrid or multistep approaches. Additionally, these methods were mostly created to solve second–order differential equations numerically. Following this, we give the following theoretical models of research, together with the associated reference lists.:

- Exponentially–fitted, trigonometrically–fitted, Phase–Fitted and Amplification–Fitted Runge–Kutta and Runge–Kutta–Nyström Methods and Runge–Kutta and Runge–Kutta–Nyström Methods with minimal phase–lag (see [10]–[15], [16], [17])
- Exponentially–fitted and trigonometrically–fitted Phase–Fitted and Amplification–Fitted Multistep Methods and Multistep Methods with minimal phase–lag(see [18], [19]–[23], [24]–[28])

In his recent publications [29] and [30], Simos laid out the theory for building first-order IVPs using multistep procedures with minimum phase–lag or phase–fitted multistep methods. More specifically, he developed his theory for calculating the phase–lag and amplification–factor of multistep approaches for first–order IVPs (Explicit [29] and Implicit [30]). Furthermore, in a recent study, Saadat et. al. [33] laid forth the theory behind the creation of backward differentiation formulae (BDF) for second–order differential equations with oscillating solutions. The fact that BDF methods are implicit and designed specifically for stiff problems is well–known. When dealing with big integration intervals, the phase–fitted and amplification–fitted approaches work well. All of the examples in the paper [33] have integration areas of $\left[0, 10\,\pi\right]$, however there were no applications with high integration areas. The behavior of the specified approaches in intervals $\left[0, 50000\,\pi\right]$ or $\left[0, 100000\,\pi\right]$ would be intriguing to see.

In this study, we will investigate the development of phase–fitted and amplification–fitted approaches for the sixth algebraic order Adams–Bashforth method. For first–order IVPs, we will additionally study the impact of phase–lag and amplification–factor derivatives on the efficiency of the approaches that are developed.

Here is how the paper is structured:

- In Section 2 we present the basic formulae for our investigation.
- In Section 3 we present the sixth algebraic order Adams–Bashforth methods, which will be the target of our investigation.
- In Section 4-10 we present the newly produced methods.
- In Section 11 we study the stability for the newly obtained algorithms.
- In Section 12 we present numerical results and conclusions for the newly introduced methods.

2. The Basic Formulae

Let us consider the general form of the Adams–Bashforth methods:

$$\Upsilon_{n+1} - \Upsilon_n = h \sum_{j=1}^{k} \left[A_{j-1}\,(\omega\, h)\;\Upsilon'_{n+1-j} \right] \tag{2.1}$$

where $A_{j-1}\,(\omega\, h)\,,\, j = 1, 2, \ldots, k$ are polynomials of $\omega\, h$ and h is the step length of the integration.

For the computation of the phase–lag, the amplification–factor and its derivatives, we will use the formulae:

$$\begin{aligned}
&\cos\,(k\,v) - \cos\Big[(k-1)\;v\Big] + \sum_{j=1}^{k-1} v\, A_{j-1}\,(v)\,\sin\Big[(k-j)\;v\Big] \\
&= -c\, v^{q+2} \Big[k^2 - (k-1)^2 - \sum_{j=1}^{k-1} A_{j-1}\,(v)\;(k-j) \Big] \Longrightarrow
\end{aligned}$$

$$-c\, v^{q+2} = \frac{\cos\,(k\,v) - \cos\,[(k-1)\;v] + v\,\sum_{j=1}^{k-1} A_{j-1}\,(v)\,\sin\,[(k-j)\;v]}{2\,k - 1 - \sum_{j=1}^{k-1} A_{j-1}\,(v)\;(k-j)} \tag{2.2}$$

This is the direct formula for the computation of the phase–lag of the multistep method (2.1). Below we will describe the procedure for the computation of the phase–lag of the method (2.1).

$$\begin{aligned}
&\sin\,[k\,v] - \sin\,[(k-1)\;v] - \sum_{j=1}^{k-1} v\, A_{j-1}\,(v)\,\cos\,[(k-j)\;v] - v\, A_{k-1} \\
&= -c\, v^{q+1} \Big[-1 - \sum_{j=1}^{k-1} v^2\, A_{j-1}\,(v)\;(k-j)^2 \Big] \Longrightarrow
\end{aligned}$$

$$-c\, v^{q+1} = \frac{\sin\,[k\,v] - \sin\,[(k-1)\;v] - v\,\sum_{j=1}^{k-1} A_{n+k-j}\,(v)\,\cos\,[(k-j)\;v] - v\, A_{k-1}}{-1 - v^2\,\sum_{j=1}^{k-1} A_{j-1}\,(v)\;(k-j)^2} \tag{2.3}$$

This is the direct formula for the computation of the amplification factor of the multistep method (2.1).

In this paper also we will examine strategies for vanishing of the derivatives of the phase–lag and the derivatives of the amplification–factor.

More specifically we will present strategies on:

- The elimination of the phase-lag's derivatives
- The elimination of the amplification–factor's derivatives
- The elimination of the phase–lag's derivative and the amplification–factor's derivatives

Remark 2.1. *The above-mentioned strategies have the main property that are applied to phase–fitted and amplification–fitted methods.*

3. The Method under Investigation

For the purposes of this paper we will use the formula (2.1) with $k = 6$, with:

$$\begin{aligned} A_0(v) &= \frac{4277}{1440}, \; A_1(v) = -\frac{2641}{480}, \; A_2(v) = \frac{4991}{720} \\ A_3(v) &= -\frac{3649}{1440}, \; A_4(v) = \frac{959}{480}, \; A_5(v) = -\frac{95}{288} \end{aligned} \tag{3.1}$$

The above method has a local truncation error given by:

$$LTE = \frac{19087}{60480} h^7 \Upsilon^{\{(7)\}}(t) + O\left(h^8\right). \tag{3.2}$$

where $\Upsilon^{\{(j)\}}$ is the j-th derivative of Υ.

Applying the formulae (2.2) and (2.3) to the above method and taking the Taylor series expansions, we get:

$$PhErr = \frac{113147}{665280} v^8 - \frac{5774921}{19958400} v^{10} + \ldots. \tag{3.3}$$

$$AF = \frac{19087}{60480} v^7 - \frac{1906717}{172800} v^9 + \ldots. \tag{3.4}$$

4. Amplification–Fitted Sixth Algebraic Order Adams–Bashforth Method with Phase–Lag of Order Six

Let us considering the method (2.1) with $k = 6$, and

$$\begin{aligned} A_1(v) &= -\frac{2641}{480}, \; A_2(v) = \frac{4991}{720} \\ A_4(v) &= \frac{959}{480}, \; A_5(v) = -\frac{95}{288}. \end{aligned} \tag{4.1}$$

4.1. Disappearing the Amplification–Factor

Applying the formula (2.3) to the above method, we obtain the following result for the amplification–factor :

$$AF = \frac{Formula_1}{36000\, v^2 A_0(v) + 5760\, v^2 A_3(v) - 34053\, v^2 + 1440}, \tag{4.2}$$

where AF denotes the Amplification Factor, and:

$$\begin{aligned} Formula_1 &= 1440\, v\, A_0(v) \cos(5v) + 1440\, v\, A_3(v) \cos(2v) - 7923\, v \cos(4v) \\ &+ 9982\, v \cos(3v) + 2877\, v \cos(v) - 1440 \sin(6v) \\ &+ 1440 \sin(5v) - 475\, v. \end{aligned} \tag{4.3}$$

Given the assumption that the amplification–factor must be zero, or that $AF = 0$, we can deduce:

$$A_0(v) = \frac{Formula_2}{1440\, v\, \cos(5\, v)}, \tag{4.4}$$

where

$$\begin{aligned} Formula_2 &= -1440\, v\, A_3(v)\, \cos(2\, v) + 7923\, v \cos(4\, v) - 9982\, v\, \cos(3\, v) \\ &- 2877\, v\, \cos(v) + 1440\, \sin(6\, v) - 1440\, \sin(5\, v) + 475\, v. \end{aligned} \tag{4.5}$$

4.2. Strategy for Reducing the Phase–Lag

Once we have the values of $A_0(v)$, we can use the direct formula from reference (2.2) to obtain the phase–lag.

$$PhErr = \frac{Formula_3}{Formula_4}, \tag{4.6}$$

where

$$\begin{aligned} Formula_3 &= 1440\, \sin(3\, v)\, v^2\, A_3(V) - 475\, \sin(5\, v)\, v^2 + 2877\, \sin(4\, v)\, v^2 \\ &+ 9982\, \sin(2\, v)\, v^2 - 7923\, \sin(v)\, v^2 - 1440\, v\, \cos(v) + 1440\, v \end{aligned} \tag{4.7}$$

$$\begin{aligned} Formula_4 &= 2880\, A_3(V)\, v\, \cos(5\, v) - 7200\, v\, A_3(V) \cos(2\, v) - 14709\, v\, \cos(5\, v) \\ &+ 39615\, v\, \cos(4\, v) - 49910\, v\, \cos(3\, v) - 14385\, v\, \cos(v) \\ &+ 7200\, \sin(6\, v) - 7200\, \sin(5\, v) + 2375\, v \end{aligned} \tag{4.8}$$

and $PhErr$ denotes the phase–lag.

Through the use of the Taylor series expansion on the formula (4.6), we can derive:

$$\begin{aligned} PhErr &= \frac{(4320\, A_3(v) + 21894)}{-4320\, A_3(v) - 29814}\, v^2 \\ &+ \frac{1}{-4320\, A_3(v) - 29814}\Big(-6480\, A_3(v) - 32841 + Formula_5\Big)\, v^4 \\ &+ \frac{Formula_6}{-4320\, A_3(v) - 29814}\, v^6 + \frac{Formula_7}{-4320\, A_3(v) - 29814}\, v^8 \\ &+ \frac{Formula_8}{-4320\, A_3(v) - 29814}\, v^{10} + \ldots, \end{aligned} \tag{4.9}$$

where $Formula_j$, $j = 5\,(1)\,8$ are given in the Appendix A.

The following equation is obtained by requesting that the phase-lag be minimized:

$$\frac{(4320\, A_3(v) + 21894)}{-4320\, A_3(v) - 29814} = 0 \Longrightarrow A_3(v) = -\frac{3649}{720}. \tag{4.10}$$

Substituting the above value of $A_3(v)$ into (4.4), we obtain:

$$A_0(v) = \frac{Formula_9}{1440\, v\, \cos(5\, v)}, \tag{4.11}$$

where

$$\begin{aligned} Formula_9 &= 7298\,v\,\cos(2\,v) + 7923\,v\,\cos(4\,v) - 9982\,v\,\cos(3\,v) \\ &- 2877\,v\,\cos(v) + 1440\,\sin(6\,v) - 1440\,\sin(5\,v) + 475\,v \end{aligned} \tag{4.12}$$

Here are some characteristics of this innovative method:

$$\begin{aligned} A_0(v) &= \frac{Formula_9}{1440\,v\,\cos(5\,v)}, \\ A_1(v) &= -\frac{2641}{480}, \\ A_2(v) &= \frac{4991}{720}, \\ A_3(v) &= -\frac{3649}{720}, \\ A_4(v) &= \frac{959}{480}, \\ A_5(v) &= -\frac{95}{288}, \\ LTE &= \frac{19087}{60480}\,h^7\left(\Upsilon^{\{(7)\}}(t) + \omega^6\,\Upsilon'(t)\right) + O\left(h^8\right) \\ PhErr &= -\frac{77723}{665280}\,v^8 - \frac{6744709}{4989600}\,v^{10} - \ldots, \\ AF &= 0, \end{aligned} \tag{4.13}$$

where $Formula_9$ is given by (4.12).

One possible way to express $A_0(v)$ is via a Taylor series expansion:

$$A_0(v) = \frac{4277}{1440} - \frac{19087}{60480}\,v^6 - \frac{9012533}{3628800}\,v^8 - \frac{3860375567\,v^{10}}{159667200} + \ldots. \tag{4.14}$$

Remark 4.1. *The resultant method will be identical to the one given above regardless of whether we select three free parameters (*$A_j(v)$, $j = 0, 2, 3$*) or four free parameters (*$A_j(v)$, $j = 0, 2\,(1)\,4$*).*

Remark 4.2. *If we select four free parameters (*$A_j(v)$, $j = 0,\ 2\,(1)\,5$*), and following the same methodology, which is mentioned above, the resultant method is the following:*

$$
\begin{aligned}
A_0(v) &= \frac{Formula_{10}}{362880\, v\, \cos(5\, v)}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= \frac{341521}{60480}, \\
A_3(v) &= -\frac{14069}{4480}, \\
A_4(v) &= \frac{336389}{362880}, \\
A_5(v) &= -\frac{41977}{362880}, \\
LTE &= -\frac{77723}{362880} h^3 \left(\Upsilon^{\{(3)\}}(t) + \omega^2\, \Upsilon'(t)\right) + O\left(h^4\right) \\
PhErr &= \frac{837043}{39916800} v^{10} + \frac{4074283125137}{15933509222400} v^{12} + \ldots, \\
AF &= 0, \qquad (4.15)
\end{aligned}
$$

where $Formula_{10}$ is given by:

$$
\begin{aligned}
Formula_{10} &= -2049126\, v\, \cos(3\, v) + 1139589\, v\, \cos(2\, v) \\
&- 336389\, \cos(v)\; v + 1996596\, v\, \cos(4\, v) \\
&+ 362880\, \sin(6\, v) - 362880\, \sin(5\, v) \\
&+ 41977\, v. \qquad (4.16)
\end{aligned}
$$

One possible way to express $A_0(v)$ is via a Taylor series expansion:

$$
\begin{aligned}
A_0(v) &= \frac{1155527}{362880} + \frac{77723}{362880} v^2 + \frac{854953}{4354560} v^4 \\
&+ \frac{1449569}{18662400} v^6 + \frac{130598021}{1463132160} v^8 + \ldots. \qquad (4.17)
\end{aligned}
$$

The above method is of second algebraic order, so it is not convergent in our numerical tests.

5. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method

Let us considering the method (2.1) with $k = 6$, and coefficients given by (4.1).

Applying the formula (2.3) to the above method, we obtain the relation (4.2) for the amplification–factor.

Applying the formula (2.2) to the above method, we obtain the following result for the phase–lag:

$$PhErr = \frac{Formula_{11}}{-14709 + 7200\, A_0\,(v) + 2880\, A_3\,(v)}, \tag{5.1}$$

where:

$$\begin{aligned} Formula_{11} &= -1440\, v\, A_0\,(v)\, \sin(5\, v) - 1440\, v\, A_3\,(v)\, \sin(2\, v) \\ &+ 7923\, v\, \sin(4\, v) - 9982\, v\, \sin(3\, v) \\ &- 2877\, v\, \sin(v) - 1440\, \cos(6\, v) + 1440\, \cos(5\, v)\,. \end{aligned} \tag{5.2}$$

Requesting the elimination of the phase–lag and the amplification–factor, i.e. requesting $PhErr = 0$, and $AF = 0$, we obtain:

$$A_0\,(v) = \frac{Formula_{12}}{\left[5760\, (\cos(v))^3 + 5760\, (\cos(v))^2 - 1440\, \cos(v) - 1440\right] v} \tag{5.3}$$

$$A_3\,(v) = \frac{Formula_{13}}{\left[2880\, (\cos(v))^3 + 2880\, (\cos(v))^2 - 720\, \cos(v) - 720\right] v}, \tag{5.4}$$

where:

$$\begin{aligned} Formula_{12} &= 11520\, \sin(v)\, (\cos(v))^3 + 5760\, \sin(v)\, (\cos(v))^2 \\ &+ 14896\, (\cos(v))^2\, v - 5760\, \sin(v)\, \cos(v) \\ &+ 7791\, v\, \cos(v) - 1440\, \sin(v) - 7105\, v \end{aligned} \tag{5.5}$$

$$\begin{aligned} Formula_{13} &= 3800\, v\, (\cos(v))^5 - 7708\, v\, (\cos(v))^4 \\ &- 14358\, (\cos(v))^3\, v - 7078\, (\cos(v))^2\, v \\ &- 29\, v\, \cos(v) - 720\, \sin(v) + 4199\, v. \end{aligned} \tag{5.6}$$

One possible way to express $A_0\,(v)$, and $A_3\,(v)$ is via a Taylor series expansion:

$$A_0\,(v) = \frac{4277}{1440} - \frac{5257}{51840}\, v^6 - \frac{5467}{48600}\, v^8 - \frac{22023953}{205286400}\, v^{10} - \ldots, \tag{5.7}$$

$$A_3\,(v) = -\frac{3649}{720} - \frac{77723}{362880}\, v^6 - \frac{665123}{5443200}\, v^8 - \frac{158957881}{1437004800}\, v^{10} - \ldots \tag{5.8}$$

Here are some characteristics of this innovative method:

$$
\begin{aligned}
A_0(v) &= \frac{Formula_{12}}{\left[5760\,(\cos(v))^3 + 5760\,(\cos(v))^2 - 1440\,\cos(v) - 1440\right] v}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= \frac{4991}{720}, \\
A_3(v) &= \frac{Formula_{13}}{\left[2880\,(\cos(v))^3 + 2880\,(\cos(v))^2 - 720\,\cos(v) - 720\right] v}, \\
A_4(v) &= \frac{959}{480}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= \frac{19087}{60480}\,h^7\left(\Upsilon^{\{(7)\}}(t) + \omega^6\,\Upsilon'(t)\right) + O\left(h^8\right) \\
PhErr &= 0, \\
AF &= 0, \qquad (5.9)
\end{aligned}
$$

where $Formula_{12}$ and $Formula_{13}$ are given by (5.5) and (5.6) respectively.

6. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method with Eliminated the First Derivative of the Phase–Lag

Let us considering the method (2.1) with $k = 6$, and

$$
\begin{aligned}
A_1(v) &= -\frac{2641}{480},\ A_4(v) = \frac{959}{480} \\
A_5(v) &= -\frac{95}{288}. \qquad (6.1)
\end{aligned}
$$

Applying the formulae (2.2) and (2.3) to the above method, we obtain the following result for the phase–lag and for the amplification–factor:

$$
PhErr = \frac{Formula_{14}}{-14885 + 2400\,A_0(v) + 1440\,A_2(v) + 960\,A_3(v)}, \qquad (6.2)
$$

where:

$$
\begin{aligned}
Formula_{14} &= -480\,v\,A_0(v)\,\sin(5\,v) - 480\,v\,A_2(v)\,\sin(3\,v) - 480\,v\,A_3(v)\,\sin(2\,v) \\
&+ 2641\,v\,\sin(4\,v) - 959\,v\,\sin(v) - 480\,\cos(6\,v) + 480\,\cos(5\,v)\,. \qquad (6.3)
\end{aligned}
$$

$$AF = \frac{Formula_{15}}{36000\, v^2\, A_0\,(v) + 12960\, v^2\, A_2\,(v) + 5760\, v^2\, A_3\,(v) - 123891\, v^2 + 1440}, \tag{6.4}$$

where:

$$\begin{aligned} Formula_{15} &= 1440\, v\, A_0\,(v)\, \cos\,(5\,v) + 1440\, v\, A_2\,(v)\, \cos\,(3\,v) \\ &+ 1440\, v\, A_3\,(v)\, \cos\,(2\,v) - 7923\, v\, \cos\,(4\,v) \\ &+ 2877\, v\, \cos\,(v) + 1440\, \sin\,(5\,v) \\ &- 1440\, \sin\,(6\,v) - 475\, v. \end{aligned} \tag{6.5}$$

Based on the formula (6.2), we obtain the formula of the derivative of the phase–lag:

$$\frac{\partial PhErr}{\partial v} = \frac{Formula_{16}}{-14885 + 2400\, A_0\,(v) + 1440\, A_2\,(v) + 960\, A_3\,(v)}, \tag{6.6}$$

where

$$\begin{aligned} Formula_{16} &= -480\, A_0\,(v)\, \sin\,(5\,v) - 2400\, v\, A_0\,(v)\, \cos\,(5\,v) \\ &- 480\, A_2\,(v)\, \sin\,(3\,v) - 1440\, v\, A_2\,(v)\, \cos\,(3\,v) \\ &- 480\, A_3\,(v)\, \sin\,(2\,v) - 960\, v\, A_3\,(v)\, \cos\,(2\,v) \\ &+ 2641\, \sin\,(4\,v) + 10564\, v \cos\,(4\,v) \\ &- 959\, \sin\,(v) - 959\, v \cos\,(v) \\ &+ 2880\, \sin\,(6\,v) - 2400\, \sin\,(5\,v)\,. \end{aligned} \tag{6.7}$$

Requesting the elimination of the phase–lag, the amplification–factor, and the derivative of the phase–lag i.e. requesting $PhErr = 0$, and $AF = 0$, and $\frac{\partial PhErr}{\partial v} = 0$ we obtain:

$$A_0\,(v) = \frac{Formula_{17}}{4\, DENOM_1}, \tag{6.8}$$

$$A_2\,(v) = -\frac{Formula_{18}}{DENOM_1}, \tag{6.9}$$

$$A_3\,(v) = \frac{Formula_{19}}{2\, DENOM_1}, \tag{6.10}$$

where

$$\begin{aligned} DENOM_1 &= 360\, v^2\, \sin\,(v)\, \Big[8\, (\cos\,(v))^3 \\ &+ 8\, (\cos\,(v))^2 - 3\, \cos\,(v) - 3\Big]\, \cos\,(v) \end{aligned} \tag{6.11}$$

and $Formula_q$, $q = 17\,(1)\,19$ are given in the Appendix B.

One possible way to express $A_0(v)$, $A_2(v)$, and $A_3(v)$ is via a Taylor series expansion:

$$\begin{aligned}
A_0(v) &= \frac{4277}{1440} + \frac{113147}{403200} v^4 + \frac{478741}{2592000} v^6 \\
&+ \frac{380090639}{2177280000} v^8 + \frac{3122008317443}{16345929600000} v^{10} + \dots, \\
A_2(v) &= \frac{4991}{720} - \frac{113147}{134400} v^4 + \frac{532561}{2016000} v^6 \\
&- \frac{21994373}{241920000} v^8 - \frac{70999994113}{908107200000} v^{10} - \dots, \\
A_3(v) &= -\frac{3649}{720} + \frac{113147}{201600} v^4 + \frac{1404509}{18144000} v^6 \\
&+ \frac{102988747}{544320000} v^8 + \frac{1813944356603}{8172964800000} v^{10} + \dots. \qquad (6.12)
\end{aligned}$$

Here are some characteristics of this innovative method:

$$\begin{aligned}
A_0(v) &= \frac{Formula_{17}}{1440\, v^2 \sin(v) \left[8\,(\cos(v))^3 + 8\,(\cos(v))^2 - 3\cos(v) - 3\right] \cos(v)}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= -\frac{Formula_{18}}{360\, v^2 \sin(v) \left[8\,(\cos(v))^3 + 8\,(\cos(v))^2 - 3\cos(v) - 3\right] \cos(v)}, \\
A_3(v) &= \frac{Formula_{19}}{720\, v^2 \sin(v) \left[8\,(\cos(v))^3 + 8\,(\cos(v))^2 - 3\cos(v) - 3\right]}, \\
A_4(v) &= \frac{959}{480}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= \frac{1}{1209600} h^7 \Big(381740\, \Upsilon^{\{(7)\}}(t) - 1018323\, \omega^4\, \Upsilon^{\{(3)\}}(t) \\
&- 636583\, \omega^6\, \Upsilon'(t)\Big) + O\left(h^8\right) \\
PhErr &= 0, \\
AF &= 0, \\
\frac{\partial PhErr}{\partial v} &= 0 \qquad (6.13)
\end{aligned}$$

where $Formula_{17}$, $Formula_{18}$ and $Formula_{19}$ are given in the Appendix B.

7. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method with Eliminated the First Derivative of the Amplification–Factor

Let us considering the method (2.1) with $k = 6$, and coefficients given by (6.1).

The formulae of the phase–lag and the amplification–factor are given by (6.2), and (6.4) respectively.

Based on the formula (6.4), we obtain the formula of the derivative of the amplification–factor:

$$\frac{\partial AF}{\partial v} = \frac{Formula_{20}}{Formula_{21}}, \tag{7.1}$$

where $Formula_p$, $p = 20, 21$ are given in the Appendix C.

Requesting the elimination of the phase–lag, the amplification–factor, and the derivative of the amplification–factor i.e. requesting $PhErr = 0$, and $AF = 0$, and $\frac{\partial AF}{\partial v} = 0$ we obtain:

$$A_0\,(v) = \frac{Formula_{22}}{1440\,v^2\left[16\,(\cos(v))^4 - 14\,(\cos(v))^2 + 1\right]\,\sin(v)}, \tag{7.2}$$

$$A_2\,(v) = -\frac{Formula_{23}}{240\,v^2\left[16\,(\cos(v))^4 - 14\,(\cos(v))^2 + 1\right]\,\sin(v)}, \tag{7.3}$$

$$A_3\,(v) = \frac{Formula_{24}}{720\,v^2\left[16\,(\cos(v))^4 - 14\,(\cos(v))^2 + 1\right]\,\sin(v)}, \tag{7.4}$$

where $Formula_r$, $r = 22\,(1)\,24$ are given in the Appendix D.

One possible way to express $A_0\,(v)$, $A_2\,(v)$, and $A_3\,(v)$ is via a Taylor series expansion:

$$\begin{aligned}
A_0\,(v) &= \frac{4277}{1440} + \frac{19087}{60480}\,v^4 + \frac{5026067}{10886400}\,v^6 \\
&+ \frac{343539137}{205286400}\,v^8 + \frac{181557803827}{24518894400}\,v^{10} + \ldots, \\
A_2\,(v) &= \frac{4991}{720} - \frac{19087}{20160}\,v^4 - \frac{1549157}{3628800}\,v^6 \\
&- \frac{1689094423}{479001600}\,v^8 - \frac{701225628113}{43589145600}\,v^{10} - \ldots, \\
A_3\,(v) &= -\frac{3649}{720} + \frac{19087}{30240}\,v^4 + \frac{231883}{388800}\,v^6 \\
&+ \frac{298991381}{102643200}\,v^8 + \frac{5168061096167}{392302310400}\,v^{10} + \ldots.
\end{aligned} \tag{7.5}$$

Here are some characteristics of this innovative method:

$$\begin{aligned}
A_0(v) &= \frac{Formula_{22}}{1440\, v^2 \left[16\, (\cos(v))^4 - 14\, (\cos(v))^2 + 1\right] \sin(v)}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= -\frac{Formula_{23}}{240\, v^2 \left[16\, (\cos(v))^4 - 14\, (\cos(v))^2 + 1\right] \sin(v)}, \\
A_3(v) &= \frac{Formula_{24}}{720\, v^2 \left[16\, (\cos(v))^4 - 14\, (\cos(v))^2 + 1\right] \sin(v)}, \\
A_4(v) &= \frac{959}{480}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= \frac{19087}{60480}\, h^7 \left(\Upsilon^{\{(7)\}}(t) - 3\,\omega^4\, \Upsilon^{\{(3)\}}(t) - 2\,\omega^6\, \Upsilon'(t)\right) + O\left(h^8\right) \\
PhErr &= 0, \\
AF &= 0, \\
\frac{\partial AF}{\partial v} &= 0
\end{aligned} \tag{7.6}$$

where $Formula_{22}$, $Formula_{23}$, and $Formula_{24}$ are given in the Appendix D.

8. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method with Eliminated the First Derivative of the Phase–Lag and the First Derivative of the Amplification–Factor

Let us considering the method (2.1) with $k = 6$, and:

$$A_1(v) = -\frac{2641}{480},\ A_5(v) = -\frac{95}{288}. \tag{8.1}$$

Applying the formulae (2.2) and (2.3) to the above method, we obtain the following result for the phase–lag and for the amplification–factor:

$$PhErr = \frac{Formula_{25}}{-15844 + 2400\, A_0(v) + 1440\, A_2(v) + 960\, A_3(v) + 480\, A_4(v)}, \tag{8.2}$$

$$AF = \frac{Formula_{26}}{DENOM_2}, \tag{8.3}$$

$$\begin{aligned}
DENOM_2 &= 36000\, v^2\, A_0(v) + 12960\, v^2\, A_2(v) + 5760\, v^2\, A_3(v) \\
&+ 1440\, v^2\, A_4(v) - 126768\, v^2 + 1440
\end{aligned} \tag{8.4}$$

Based on the formulae (8.2), and (8.3) we obtain the formulae of the derivatives of the phase–lag and the amplification–factor:

$$\frac{\partial PhErr}{\partial v} = \frac{Formula_{27}}{-15844 + 2400\, A_0\,(v) + 1440\, A_2\,(v) + 960\, A_3\,(v) + 480\, A_4\,(v)}, \tag{8.5}$$

$$\frac{\partial AF}{\partial v} = \frac{Formula_{28}}{Formula_{29}}, \tag{8.6}$$

where $Formula_s$, $s = 25\,(1)\,29$ are given in the Appendix E.

Requesting the elimination of the phase–lag, the amplification–factor, and the derivative of the phase–lag and the amplification–factor i.e. requesting $PhErr = 0$, and $AF = 0$, $\frac{\partial PhErr}{\partial v} = 0$, and $\frac{\partial AF}{\partial v} = 0$ we obtain:

$$A_0\,(v) = \frac{Formula_{30}}{720\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \tag{8.7}$$

$$A_2\,(v) = -\frac{Formula_{31}}{360\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \tag{8.8}$$

$$A_3\,(v) = -\frac{Formula_{32}}{720\,\sin(v)\,v^2\,(\cos(v)+1)}, \tag{8.9}$$

$$A_4\,(v) = \frac{Formula_{33}}{720\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \tag{8.10}$$

where $Formula_t$, $t = 30\,(1)\,33$ are given in the Appendix F.

One possible way to express $A_0\,(v)$, $A_2\,(v)$, $A_3\,(v)$, and $A_4\,(v)$ is via a Taylor series expansion:

$$\begin{aligned}
A_0\,(v) &= \frac{4277}{1440} + \frac{5257}{23040}\,v^4 + \frac{43519}{518400}\,v^6 \\
&+ \frac{1884449}{54743040}\,v^8 + \frac{4658893}{333590400}\,v^{10} + \ldots, \\
A_2\,(v) &= \frac{4991}{720} - \frac{34049}{80640}\,v^4 + \frac{1599299}{3628800}\,v^6 \\
&+ \frac{29768581}{958003200}\,v^8 + \frac{4094442361}{130767436800}\,v^{10} + \ldots, \\
A_3\,(v) &= -\frac{3649}{720} - \frac{275}{4032}\,v^4 + \frac{1625}{72576}\,v^6 \\
&- \frac{15955}{9580032}\,v^8 + \frac{715175}{5230697472}\,v^{10} + \ldots, \\
A_4\,(v) &= \frac{959}{480} + \frac{42299}{161280}\,v^4 + \frac{152629}{1814400}\,v^6 \\
&+ \frac{13262743}{383201280}\,v^8 + \frac{140694937}{10059033600}\,v^{10} + \ldots.
\end{aligned} \tag{8.11}$$

Here are some characteristics of this innovative method:

$$
\begin{aligned}
A_0(v) &= \frac{Formula_{30}}{720\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= -\frac{Formula_{31}}{360\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \\
A_3(v) &= -\frac{Formula_{32}}{720\,\sin(v)\,v^2\,(\cos(v)+1)}, \\
A_4(v) &= \frac{Formula_{33}}{720\,\sin(v)\,v^2\,(\cos(v)+1)\,\cos(v)}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= \frac{19087}{60480}\,h^7\left(\Upsilon^{\{(7)\}}(t) - 3\,\omega^4\,\Upsilon^{\{(3)\}}(t) - 2\,\omega^6\,\Upsilon'(t)\right) + O\left(h^8\right) \\
PhErr &= 0, \\
AF &= 0, \\
\frac{\partial PhErr}{\partial v} &= 0, \\
\frac{\partial AF}{\partial v} &= 0
\end{aligned}
\tag{8.12}
$$

where $Formula_{30}$, $Formula_{31}$, $Formula_{32}$, and $Formula_{33}$ are given in the Appendix F.

9. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method with Eliminated the First and Second Derivatives of the Phase–Lag

Let us considering the method (2.1) with $k = 6$ and coefficients given by (8.1).

Applying the formulae (2.2) and (2.3) to the above method, we obtain the formulae for the phase–lag and for the amplification–factor given by (8.2), and (8.3). Also, the first derivative of the phase–lag is given by the formula (8.5). For the second derivative of the phase–lag we have the following formula:

$$
\frac{\partial^2 PhErr}{\partial v^2} = \frac{Formula_{34}}{-3961 + 600\,A_0(v) + 360\,A_2(v) + 240\,A_3(v) + 120\,A_4(v)},
\tag{9.1}
$$

where:

$$\begin{aligned} Formula_{34} &= 3000\,v\,A_0\left(v\right)\,\sin\left(5\,v\right)+1080\,v\,A_2\left(v\right)\,\sin\left(3\,v\right) \\ &+ 480\,v\,A_3\left(v\right)\,\sin\left(2\,v\right)+120\,v\,A_4\left(v\right)\,\sin\left(v\right) \\ &- 1200\,\cos\left(5\,v\right)A_0\left(v\right)-10564\,v\,\sin\left(4\,v\right) \\ &- 720\,A_2\left(v\right)\,\cos\left(3\,v\right)-480\,A_3\left(v\right)\,\cos\left(2\,v\right) \\ &- 240\,\cos\left(v\right)A_4\left(v\right)+4320\,\cos\left(6\,v\right) \\ &- 3000\,\cos\left(5\,v\right)+5282\,\cos\left(4\,v\right). \end{aligned} \tag{9.2}$$

Requesting the elimination of the phase–lag, the amplification–factor, and the first and second derivatives of the phase–lag i.e. requesting $PhErr = 0$, and $AF = 0$, $\frac{\partial PhErr}{\partial v} = 0$, and $\frac{\partial^2 PhErr}{\partial v^2} = 0$ we obtain:

$$A_0\left(v\right) = \frac{Formula_{35}}{Formula_{39}}, \tag{9.3}$$

$$A_2\left(v\right) = 3\,\frac{Formula_{36}}{Formula_{39}}, \tag{9.4}$$

$$A_3\left(v\right) = \frac{Formula_{37}}{Formula_{40}}, \tag{9.5}$$

$$A_4\left(v\right) = \frac{Formula_{38}}{Formula_{39}}, \tag{9.6}$$

where $Formula_w$, $w = 35\,(1)\,40$ are given in the Appendix G.

One possible way to express $A_0\left(v\right)$, $A_2\left(v\right)$, $A_3\left(v\right)$, and $A_4\left(v\right)$ is via a Taylor series expansion:

$$\begin{aligned} A_0\left(v\right) &= \frac{4277}{1440}-\frac{113147}{524160}\,v^2+\frac{108511727}{408844800}\,v^4 \\ &+ \frac{6336194393}{47834841600}\,v^6+\frac{1734993128041}{24874117632000}\,v^8 \\ &+ \frac{816521244319919}{23903032079646720}\,v^{10}+\ldots, \\ A_2\left(v\right) &= \frac{4991}{720}+\frac{113147}{65520}\,v^2-\frac{676747223}{204422400}\,v^4 \\ &+ \frac{1303972693}{956696832}\,v^6+\frac{826965704731}{12437058816000}\,v^8 \\ &+ \frac{9541079356137389}{74696975248896000}\,v^{10}+\ldots, \end{aligned}$$

$$\begin{aligned}
A_3(v) &= -\frac{3649}{720} - \frac{113147}{43680} v^2 + \frac{87351361}{17035200} v^4 \\
&- \frac{25519210309}{7972473600} v^6 + \frac{2322145979029}{3109264704000} v^8 \\
&- \frac{23224935963698753\, v^{10}}{149393950497792000} - \ldots, \\
A_4(v) &= \frac{959}{480} + \frac{113147}{104832} v^2 - \frac{26997625}{16353792} v^4 \\
&+ \frac{34208535173}{47834841600} v^6 + \frac{116076749311}{1658274508800} v^8 \\
&+ \frac{16930901745402511}{199191933997056000} v^{10} + \ldots. \qquad (9.7)
\end{aligned}$$

Here are some characteristics of this innovative method:

$$\begin{aligned}
A_0(v) &= \frac{Formula_{35}}{Formula_{39}}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= 3\,\frac{Formula_{36}}{Formula_{39}}, \\
A_3(v) &= \frac{Formula_{37}}{Formula_{40}}, \\
A_4(v) &= \frac{Formula_{38}}{Formula_{39}}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= -\frac{113147}{262080} h^5 \left(\omega^2\, \Upsilon^{\{(3)\}}(t) + \omega^4\, \Upsilon'(t)\right) + O\left(h^6\right) \\
PhErr &= 0, \\
AF &= 0, \\
\frac{\partial PhErr}{\partial v} &= 0, \\
\frac{\partial^2 PhErr}{\partial v^2} &= 0 \qquad (9.8)
\end{aligned}$$

where $Formula_w$, $w = 35\,(1)\,40$ are given in the Appendix G.

10. Phase–Fitted and Amplification–Fitted Sixth Algebraic Order Adams-Bashforth Method with Eliminated the First and Second Derivatives of the Amplification–Factor

Let us considering the method (2.1) with $k = 6$ and coefficients given by (8.1).

Applying the formulae (2.2) and (2.3) to the above method, we obtain the formulae for the phase–lag and for the amplification–factor given by (8.2), and (8.3). Also, the first derivative of the amplification–factor is given by the formula (8.6). For the second derivative of the amplification–factor we have the following formula:

$$\frac{\partial^2 AF}{\partial v^2} = \frac{Formula_{41}}{Formula_{42}}, \tag{10.1}$$

where $Formula_{41}$, and $Formula_{42}$ are given in the Appendix H.

Requesting the elimination of the phase–lag, the amplification–factor, and the first and second derivatives of the amplification–factor i.e. requesting $PhErr = 0$, and $AF = 0$, $\frac{\partial AF}{\partial v} = 0$, and $\frac{\partial^2 AF}{\partial v^2} = 0$ we obtain:

$$\begin{aligned}
A_0\,(v) &= \frac{Formula_{43}}{Formula_{44}}, && (10.2)\\
A_2\,(v) &= \frac{Formula_{45}}{Formula_{44}}, && (10.3)\\
A_3\,(v) &= \frac{Formula_{46}}{Formula_{47}}, && (10.4)\\
A_4\,(v) &= \frac{Formula_{48}}{Formula_{44}}, && (10.5)
\end{aligned}$$

where $Formula_j$, $j = 43\,(1)\,48$ are given in the Appendix I.

One possible way to express $A_0\,(v)$, $A_2\,(v)$, $A_3\,(v)$, and $A_4\,(v)$ is via a Taylor series expansion:

$$\begin{aligned}
A_0\,(v) &= \frac{4277}{1440} - \frac{19087}{221760}\,v^2 + \frac{75336419}{146361600}\,v^4\\
&+ \frac{8081150717}{28979596800}\,v^6 - \frac{8902601992859}{55254431232000}\,v^8\\
&- \frac{16253849764808753}{25527547229184000}\,v^{10} + \ldots,\\
A_2\,(v) &= \frac{4991}{720} + \frac{19087}{36960}\,v^2 - \frac{9537141}{2710400}\,v^4\\
&+ \frac{8758381411}{2414966400}\,v^6 + \frac{113485123835807}{27627215616000}\,v^8\\
&+ \frac{4362959009474167}{1418197068288000}\,v^{10} + \ldots,
\end{aligned}$$

$$\begin{aligned}
A_3(v) &= -\frac{3649}{720} - \frac{19087}{27720}\,v^2 + \frac{21196121}{4573800}\,v^4 \\
&- \frac{1542934573}{226403100}\,v^6 - \frac{2661894553001}{647512866000}\,v^8 \\
&- \frac{100985363074237}{42735849156000}\,v^{10} - \ldots, \\
A_4(v) &= \frac{959}{480} + \frac{19087}{73920}\,v^2 - \frac{26511853}{16262400}\,v^4 \\
&+ \frac{111360253}{42932736}\,v^6 + \frac{520892011638311}{165763293696000}\,v^8 \\
&+ \frac{32119550025915773}{15316528337510400}\,v^{10} + \ldots.
\end{aligned} \tag{10.6}$$

Here are some characteristics of this innovative method:

$$\begin{aligned}
A_0(v) &= \frac{Formula_{43}}{Formula_{44}}, \\
A_1(v) &= -\frac{2641}{480}, \\
A_2(v) &= \frac{Formula_{45}}{Formula_{44}}, \\
A_3(v) &= \frac{Formula_{46}}{Formula_{47}}, \\
A_4(v) &= \frac{Formula_{48}}{Formula_{44}}, \\
A_5(v) &= -\frac{95}{288}, \\
LTE &= \frac{19087}{55440}\,h^6\left(\omega^2\,\Upsilon^{\{(4)\}}(t) + \omega^4\,\Upsilon^{\{(2)\}}(t)\right) + O\left(h^7\right) \\
PhErr &= 0, \\
AF &= 0, \\
\frac{\partial AF}{\partial v} &= 0, \\
\frac{\partial^2 AF}{\partial v^2} &= 0
\end{aligned} \tag{10.7}$$

where $Formula_j$, $j = 43\,(1)\,48$ are given in the Appendix I.

11. Stability Analysis

11.1. Seven–Step Adams–Bashforth Methods

Consider the seven–step Adams–Bashforth methods given by (2.1).

Utilizing the (2.1) algorithm to solve the scalar test equation

$$\Upsilon' = \lambda\,\Upsilon \quad \text{where} \quad \lambda \in \mathcal{C}, \tag{11.1}$$

we obtain the following difference equation

$$\Upsilon_{n+1} - \sum_{j=1}^{6} \left[F_{j-1}\,(W)\ \Upsilon_{n+1-j} \right] = 0 \tag{11.2}$$

where $W = \lambda\, h$ and

$$\begin{aligned} F_0\,(W) &= 1 + A_0\, W,\ F_1\,(W) = A_1\, W,\ F_2\,(W) = A_2\, W, \\ F_3\,(W) &= A_3\, W,\ F_4\,(W) = A_4\, W,\ F_5\,(W) = A_5\, W. \end{aligned} \tag{11.3}$$

The characteristic equation of (11.2) is given by

$$q^6 - \sum_{j=1}^{6} \left[F_{j-1}\,(W)\ q^{6-j} \right] = 0. \tag{11.4}$$

We can draw the stability regions for $\zeta \in [0, 2\,\pi]$ if we solve the above equation in W and substitute $q = \exp(i\,\zeta)$ for $i = \sqrt{-1}$. The stability zone for the algorithms derived from sections 3–10 is displayed in Figures 1–8. We show the stability areas for $W = 1$, $W = 5$, $W = 10$, and $W = 100$ for these approaches.

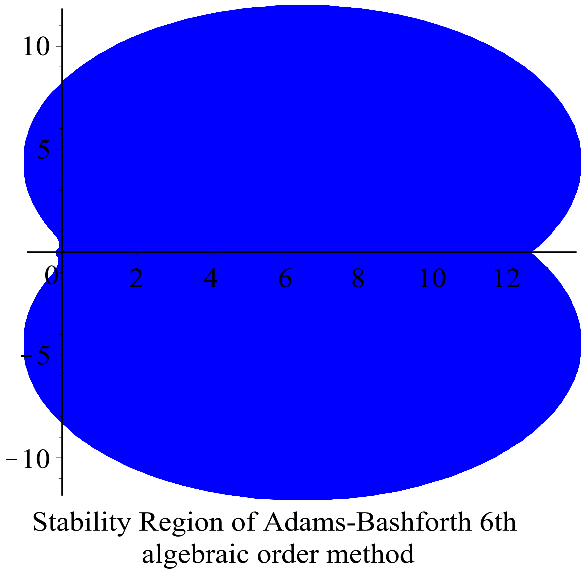
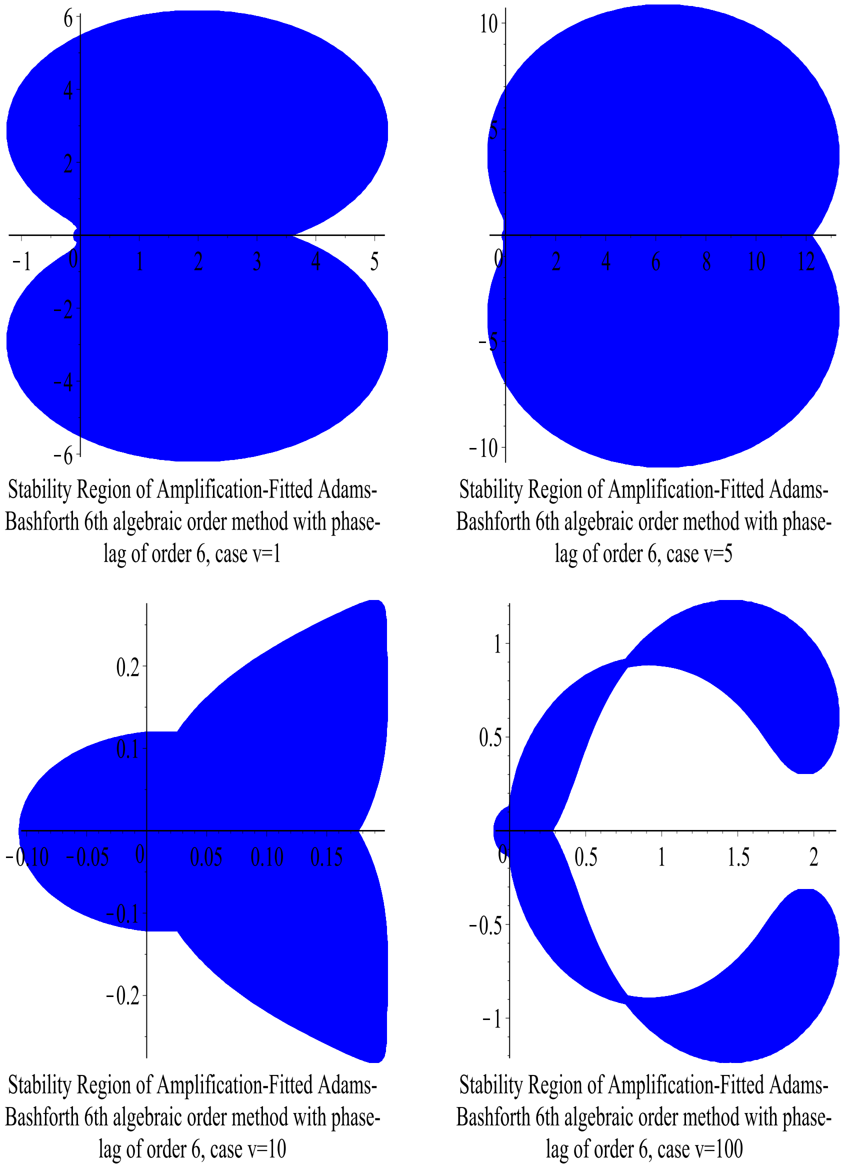


FIGURE 1. Stability region for the classical 6th Order Adams-Bashforth Method presented in Section 3



Stability Region of Amplification-Fitted Adams-Bashforth 6th algebraic order method with phase-lag of order 6, case v=1

Stability Region of Amplification-Fitted Adams-Bashforth 6th algebraic order method with phase-lag of order 6, case v=5

Stability Region of Amplification-Fitted Adams-Bashforth 6th algebraic order method with phase-lag of order 6, case v=10

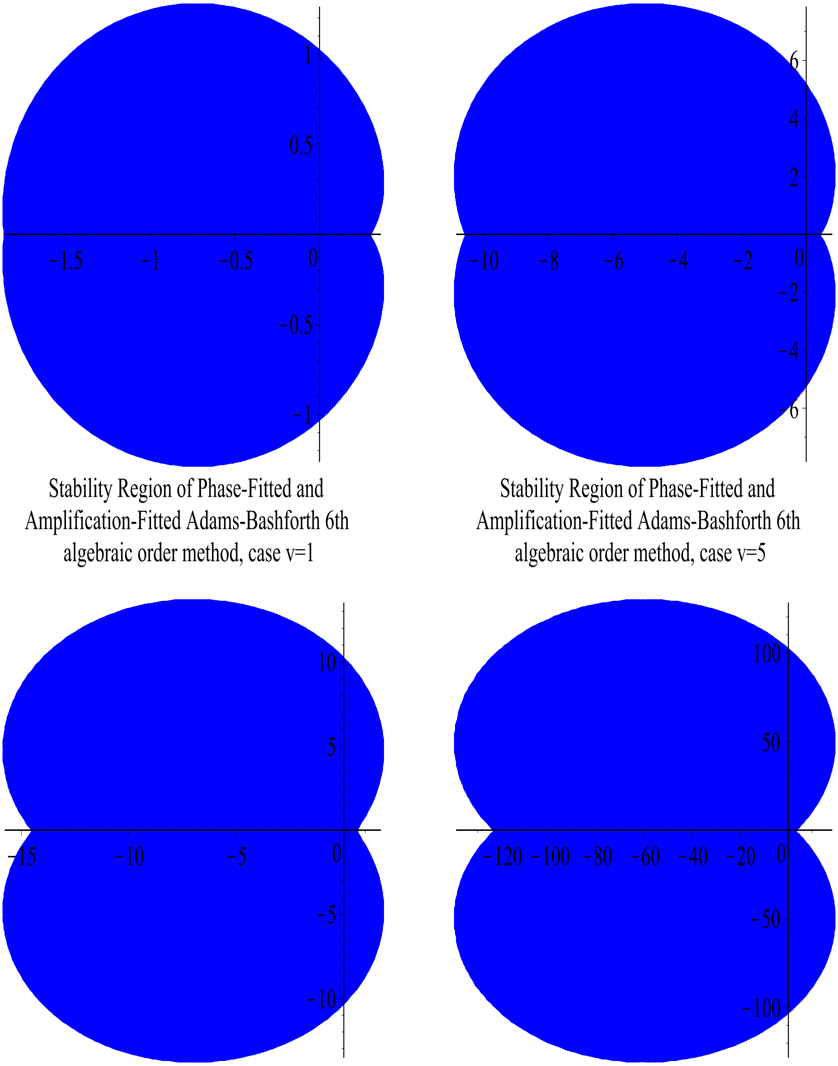
Stability Region of Amplification-Fitted Adams-Bashforth 6th algebraic order method with phase-lag of order 6, case v=100

FIGURE 2. Stability region for the Adams-Bashforth Method developed in Section 4

12. Numerical Results

12.1. Problem of Stiefel and Bettis

In their study, Stiefel and Bettis [34] examined a nearly periodic orbit problem that we also consider:



Stability Region of Phase-Fitted and Amplification-Fitted Adams-Bashforth 6th algebraic order method, case v=1

Stability Region of Phase-Fitted and Amplification-Fitted Adams-Bashforth 6th algebraic order method, case v=5

Stability Region of Phase-Fitted and Amplification-Fitted Adams-Bashforth 6th algebraic order method, case v=10

Stability Region of Phase-Fitted and Amplification-Fitted Adams-Bashforth 6th algebraic order method, case v=100

Figure 3. Stability region for the Adams-Bashforth Method developed in Section 5

$$\begin{aligned} \Upsilon_1''(t) &= -\Upsilon_1(t) + 0.001\cos(t), \quad \Upsilon_1(0) = 1, \quad \Upsilon_1'(0) = 0, \\ \Upsilon_2''(t) &= -\Upsilon_2(t) + 0.001\sin(t), \quad \Upsilon_2(0) = 0, \quad \Upsilon_2'(0) = 0.9995 \end{aligned} \tag{12.1}$$

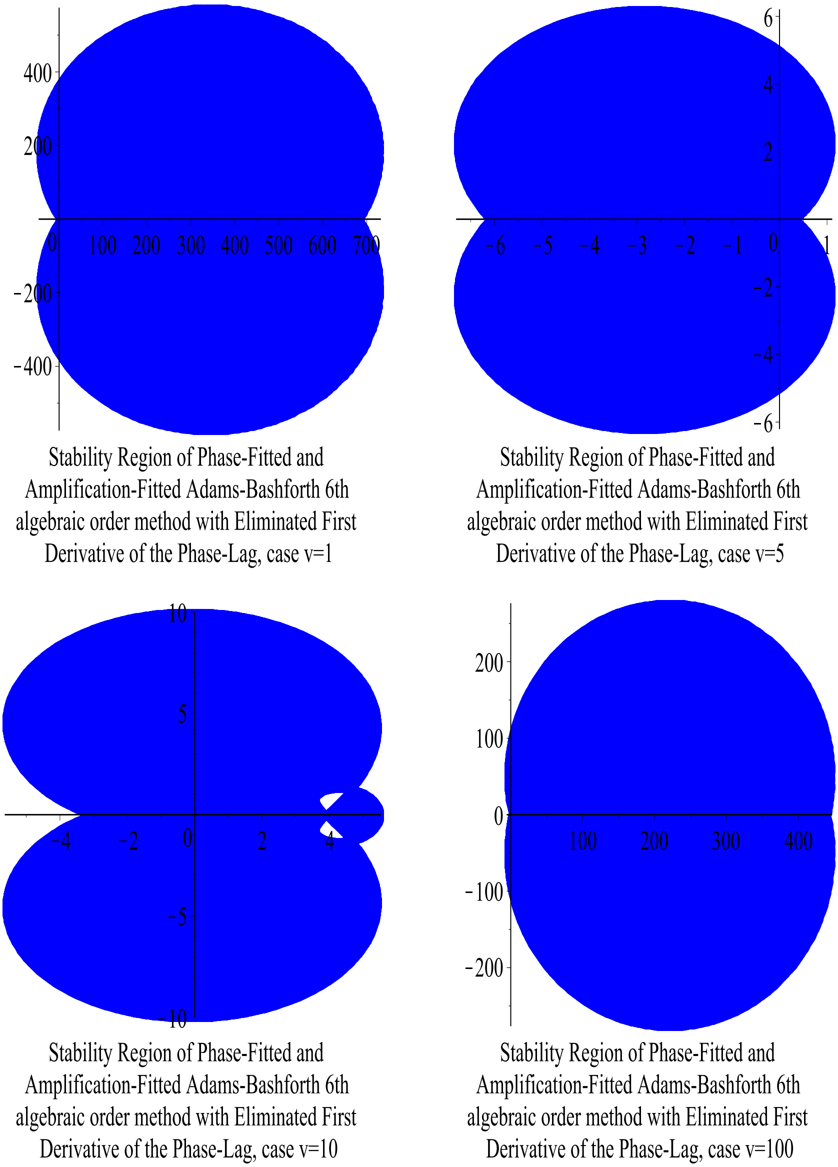


FIGURE 4. Stability region for the Adams-Bashforth Method developed in Section 6

The exact solution is

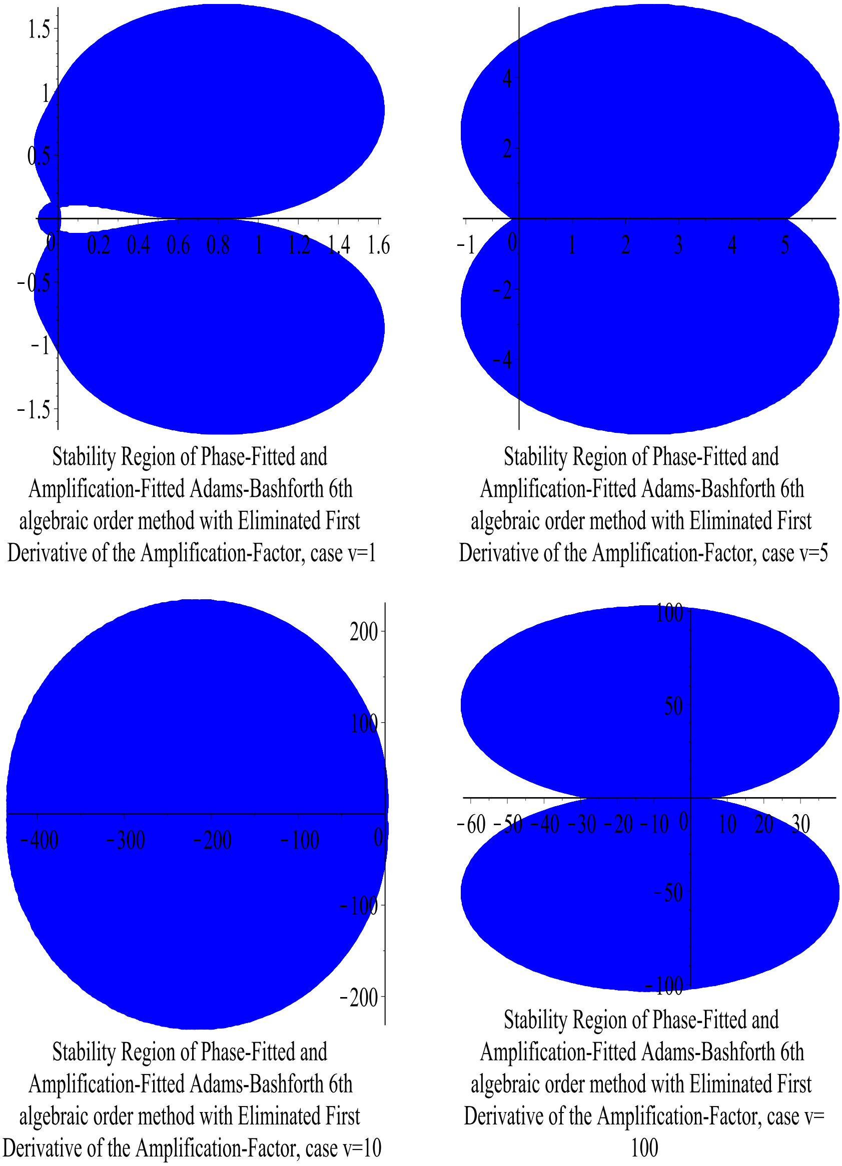


FIGURE 5. Stability region for the Adams-Bashforth Method developed in Section 7

$$\Upsilon_1(t) = \cos(t) + 0.0005\, t \sin(t),$$
$$\Upsilon_2(t) = \sin(t) - 0.0005\, t \cos(t). \tag{12.2}$$

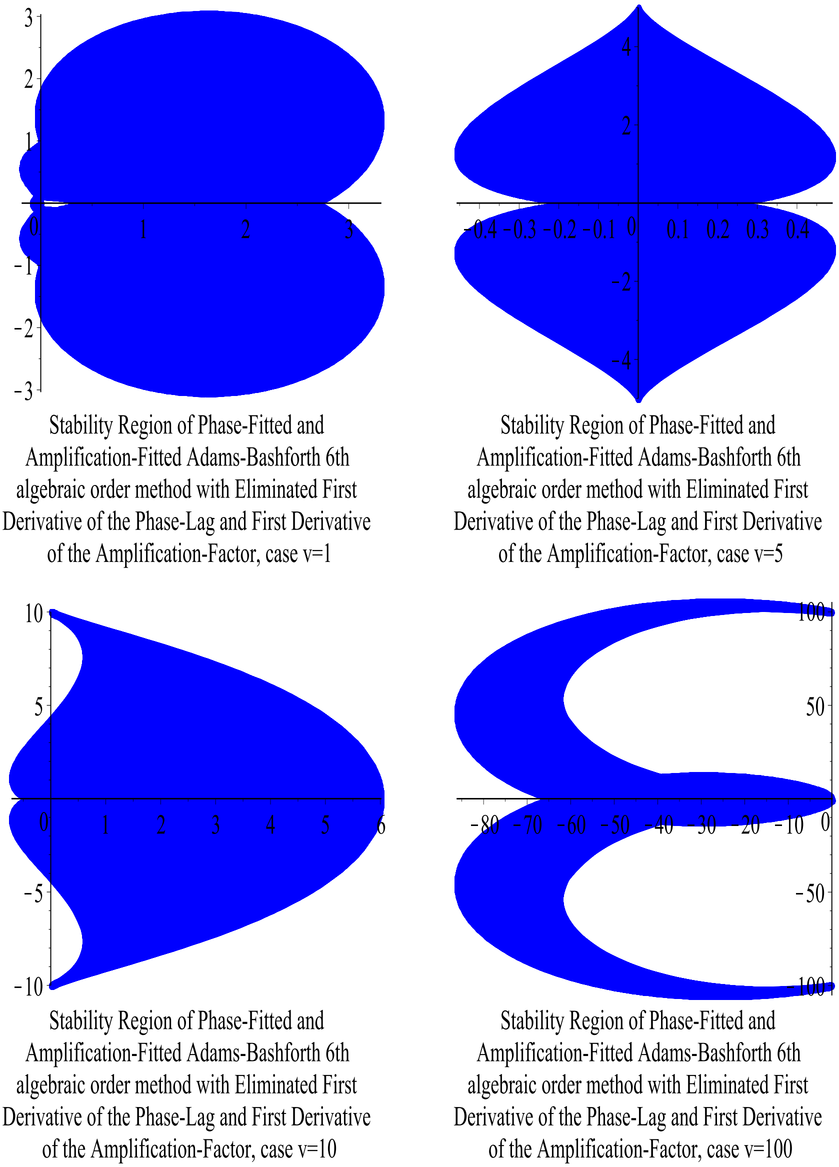


FIGURE 6. Stability region for the Adams-Bashforth Method developed in Section 8

For this problem, we use $\omega = 1$.

For $0 \leq t \leq 100000$, numerical solutions to the problem (12.1) have been established by utilizing the corresponding methods:

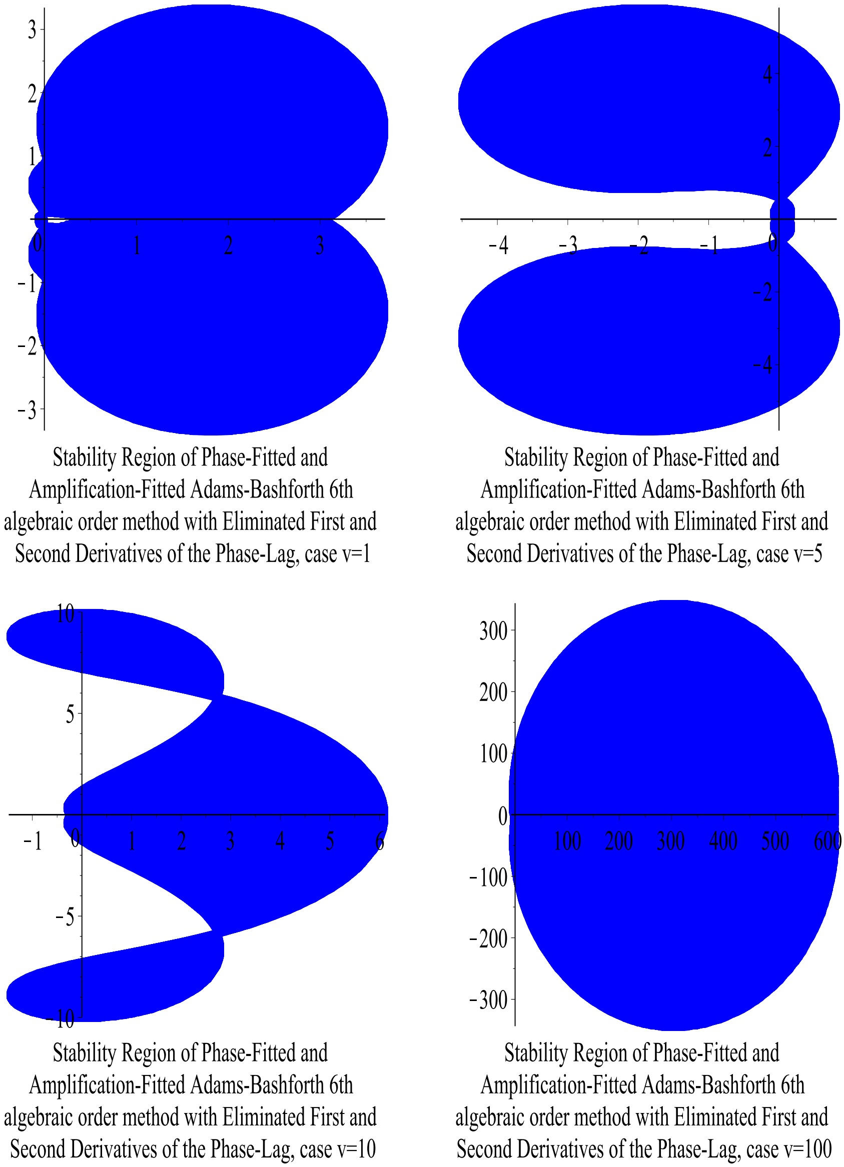


FIGURE 7. Stability region for the Adams-Bashforth Method developed in Section 9

- The Classical Adams–Bashforth Method of the sixth algebraic order, which is mentioned as **Numer. Algor. I**
- The Runge–Kutta Dormand and Prince fourth order method [16], which is mentioned as **Numer. Algor. II**

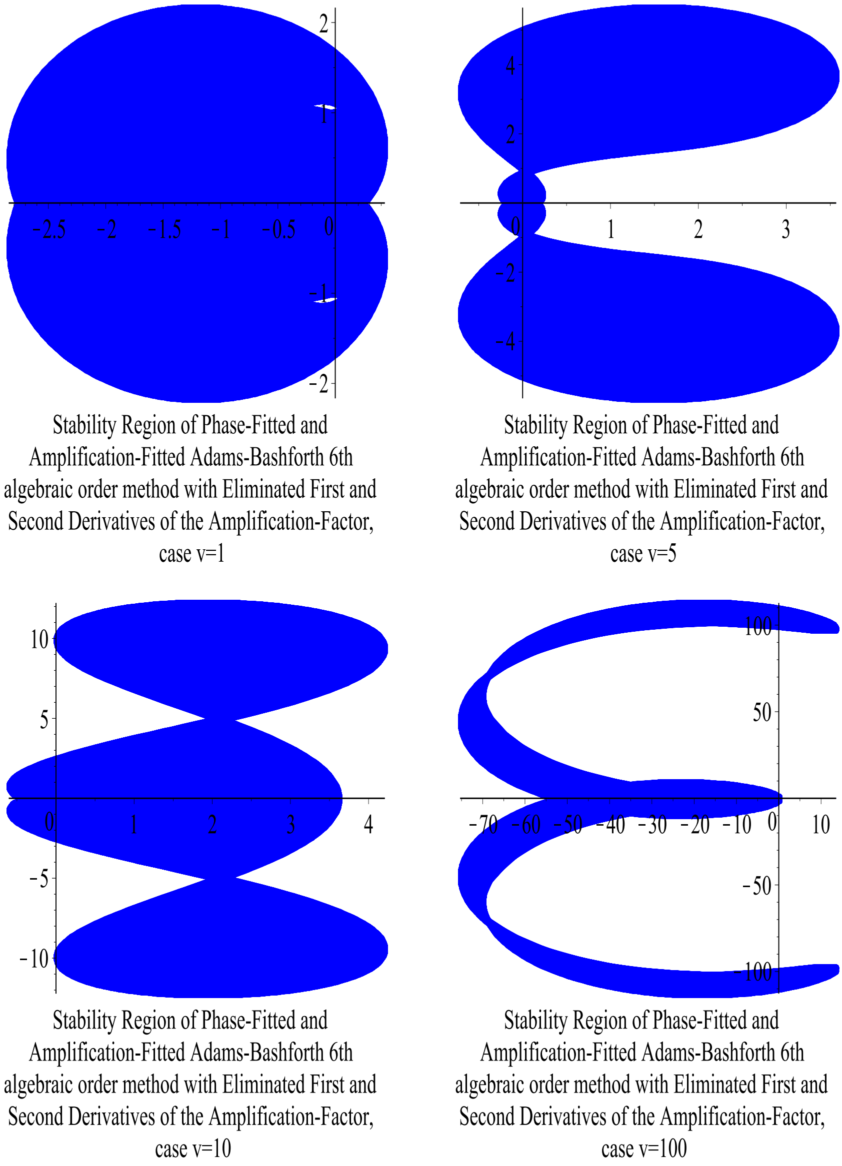


FIGURE 8. Stability region for the Adams-Bashforth Method developed in Section 10

- The Runge–Kutta Dormand and Prince fifth order method [16], which is mentioned as **Numer. Algor. III**
- The Runge–Kutta Fehlberg fourth order method [36], which is mentioned as **Numer. Algor. IV**

- The Runge–Kutta Fehlberg fifth order method [36], which is mentioned as **Numer. Algor. V**
- The Runge–Kutta Cash and Karp fifth order method [35], which is mentioned as **Numer. Algor. VI**
- The Amplification–Fitted Adams–Bashforth Method sixth algebraic order method which phase–lag of order six which is developed in Section 4, and is mentioned as **Numer. Algor. VII**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method with Eliminated the First Derivative of the Phase–Lag which is developed in section 6, and is mentioned as **Numer. Algor. VIII**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method with Eliminated the First Derivative of the Amplification–Factor which is developed in section 7, and is mentioned as **Numer. Algor. IX**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method with Eliminated the First and Second Derivatives of the Phase–Lag which is developed in section 9, and is mentioned as **Numer. Algor. X**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method with Eliminated the First and Second Derivatives of the Amplification–Factor which is developed in section 10, and is mentioned as **Numer. Algor. XI**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method which is developed in section 5, and is mentioned as **Numer. Algor. XII**
- The Phase–Fitted and Amplification–Fitted Adams–Bashforth Sixth Algebraic Order Method with Eliminated the First Derivative of the Phase–Lag and Eliminated the First Derivative of the Amplification–Factor which is developed in section 8, and is mentioned as **Numer. Algor. XIII**

The highest absolute error of the solution obtained by each of the numerical approaches outlined earlier for the Stiefel and Bettis [34] problem is shown in Figure 9.

Based on the data shown in Figure 9, we can observe the following:

- Numer. Algor. VI is more efficient than Numer. Algor. III.
- Numer. Algor. V is more efficient than the Numer. Algor. VI.
- Numer. Algor. IV and Numer. Algor. V give approximately the same accuracy.
- Numer. Algor. II is more efficient than the Numer. Algor. IV and Numer. Algor. V.
- Numer. Algor. I is more efficient than the Numer. Algor. II.
- Numer. Algor. X gives mixed results. For big step sizes is more efficient than Numer. Algor. I. For small step sizes is less efficient Numer. Algor. I.

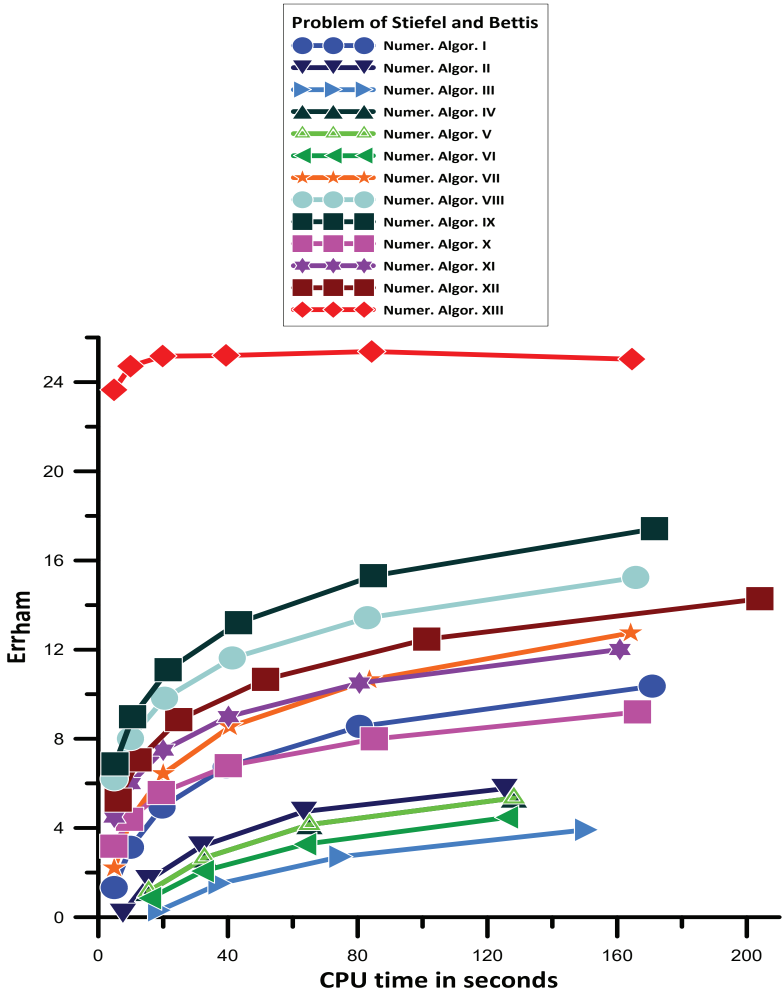


FIGURE 9. Numerical results for the problem of Stiefel and Bettis [34]

- Numer. Algor. VII is more efficient than the Numer. Algor. I.
- Numer. Algor. XI gives mixed results. For big step sizes is more efficient than Numer. Algor. VII. For small step sizes is less efficient Numer. Algor. VII.
- Numer. Algor. XII is more efficient than the Numer. Algor. XI and Numer. Algor. VII.
- Numer. Algor. VIII is more efficient than the Numer. Algor. XII.
- Numer. Algor. IX is more efficient than the Numer. Algor. VIII.
- Finally, Numer. Algor. XIII, is the most efficient one.

12.2. Problem of Franco and Palacios [37]

The following problem was examined by Franco and Palacios [37] and is taken into account here:

$$\begin{aligned} \Upsilon_1''(t) &= -\Upsilon_1(t) + \varepsilon\,\cos(\vartheta\,t), \quad \Upsilon_1(0) = 1, \quad \Upsilon_1'(0) = 0, \\ \Upsilon_2''(t) &= -\Upsilon_2(t) + \varepsilon\,\sin(\vartheta\,t), \quad \Upsilon_2(0) = 0, \quad \Upsilon_2'(0) = 1. \end{aligned} \tag{12.3}$$

The exact solution is

$$\begin{aligned} \Upsilon_1(t) &= \frac{1-\varepsilon-\vartheta^2}{1-\vartheta^2}\,\cos(t) + \frac{\varepsilon}{1-\vartheta^2}\,\cos(\vartheta\,t), \\ \Upsilon_2(t) &= \frac{1-\varepsilon\,\vartheta-\vartheta^2}{1-\vartheta^2}\,\sin(t) + \frac{\varepsilon}{1-\vartheta^2}\,\sin(\vartheta\,t), \end{aligned} \tag{12.4}$$

where $\varepsilon = 0.9$ and $\vartheta = 0.9$. For this problem, we use $\omega = \max\Big(1, |\vartheta|\Big)$.

For $0 \leq t \leq 1000000$, numerical solutions to the problem (12.3) have been established by utilizing the corresponding methods described in section 12.1.

Based on the data shown in Figure 10, we can observe the following:

- Numer. Algor. IV is more efficient than Numer. Algor. II.
- Numer. Algor. VI is more efficient than the Numer. Algor. IV.
- Numer. Algor. I gives mixed results. For big step sizes is less efficient than Numer. Algor. VI. For small step sizes is more efficient Numer. Algor. VI.
- Numer. Algor. V is more efficient than the Numer. Algor. I.
- Numer. Algor. VII gives mixed results. For big step sizes is less efficient than Numer. Algor. V. For small step sizes is more efficient Numer. Algor. V.
- Numer. Algor. III gives mixed results. For big step sizes is more efficient than Numer. Algor. VII. For small step sizes is less efficient Numer. Algor. VII.
- Numer. Algor. X gives mixed results. For big step sizes is more efficient than Numer. Algor. III. For small step sizes is less efficient Numer. Algor. III.
- Numer. Algor. XI is more efficient than the Numer. Algor. X, Numer. Algor. VII, and Numer. Algor. III.
- Numer. Algor. XII is more efficient than the Numer. Algor. XI.
- Numer. Algor. IX is more efficient than the Numer. Algor. XII.
- Numer. Algor. XIII is more efficient than the Numer. Algor. IX.
- Finally, Numer. Algor. VIII, is the most efficient one.

12.3. Nonlinear Problem of Petzold [41]

What follows is an analysis of the problem that Petzold [41] considered:

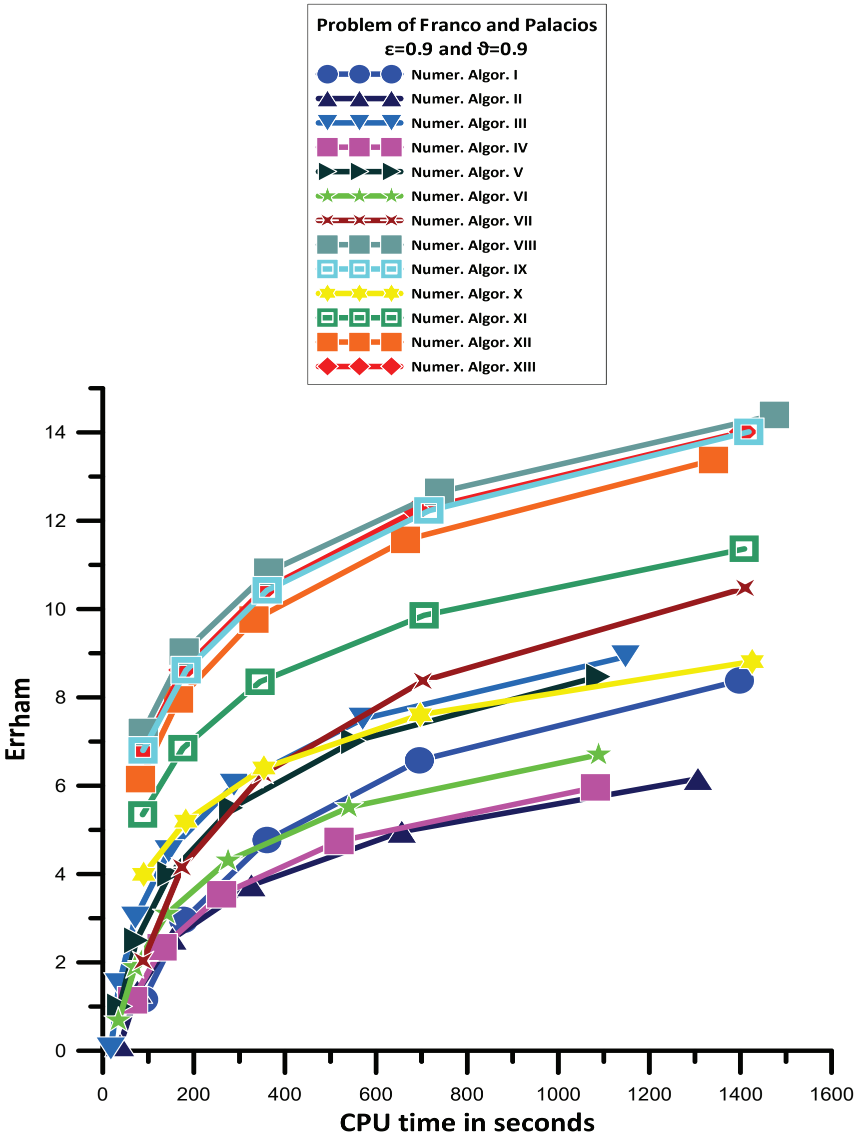


FIGURE 10. Numerical results for the problem of Franco and Palacios [37]

$$\begin{aligned} \Upsilon_1'(t) &= \lambda\, \Upsilon_2(t), \quad \Upsilon_1(0) = 1, \\ \Upsilon_2'(t) &= -\lambda\, \Upsilon_1(t) + \frac{\alpha}{\lambda}\, \sin(\lambda\, x), \quad \Upsilon_2(0) = -\frac{\alpha}{2\,\lambda^2}. \end{aligned} \tag{12.5}$$

The exact solution is

$$\Upsilon_1(t) = \Big(1 - \frac{\alpha}{2\,\lambda}\,t\Big)\,\cos(\lambda\,t),$$
$$\Upsilon_2(t) = -\Big(1 - \frac{\alpha}{2\,\lambda}\,t\Big)\,\sin(\lambda\,t) - \frac{\alpha}{2\,\lambda^2}\,\cos(\lambda\,x), \tag{12.6}$$

where $\lambda = 1000$, $\alpha = 100$. For this problem, we use $\omega = 1000$.

For $0 \le t \le 1000$, numerical solutions to the problem (12.5) have been established by utilizing the corresponding methods described in section 12.1.

The information in Figure 11 allows us to see the following:

- Numer. Algor. IV is more efficient than Numer. Algor. III.
- Numer. Algor. VI is more efficient than the Numer. Algor. IV.
- Numer. Algor. V is more efficient than the Numer. Algor. VI.
- Numer. Algor. II is more efficient than the Numer. Algor. V.
- Numer. Algor. I is more efficient than the Numer. Algor. II.
- Numer. Algor. X gives mixed results. For big step sizes is more efficient than Numer. Algor. I. For small step sizes has the same, approximately, accuracy with Numer. Algor. I.
- Numer. Algor. VII is more efficient than the Numer. Algor. I and Numer. Algor. X
- Numer. Algor. XI is more efficient than the Numer. Algor. VII.
- Numer. Algor. XII is more efficient than the Numer. Algor. XI.
- Numer. Algor. VIII is more efficient than the Numer. Algor. XII.
- Numer. Algor. IX is more efficient than the Numer. Algor. VIII.
- Finally, Numer. Algor. XIII, is the most efficient one.

12.4. Problem of Franco and G*ó*mez [39]

In their investigation, Franco and Gómez [39] considered the following problem:

$$\begin{aligned}
\Upsilon_1''(t) &= -199\,\Upsilon_1(t) - 198\,\Upsilon_2(t) + \Big(\Upsilon_1(t) + \Upsilon_2(t)\Big)^2 + \sin^2(10\,t) - 1,\\
\Upsilon_1(0) &= 2, \quad \Upsilon_1'(0) = -\varepsilon,\\
\Upsilon_2''(t) &= 99\,\Upsilon_1(t) + 98\,\Upsilon_2(t) + \Big(\Upsilon_1(t) + 2\,\Upsilon_2(t)\Big)^2 + \varepsilon^2\,\cos^2(t) - \varepsilon^2,\\
\Upsilon_2(0) &= -1, \quad \Upsilon_2'(0) = \varepsilon.
\end{aligned} \tag{12.7}$$

The exact solution is

$$\Upsilon_1(t) = 2\,\cos(10\,t) - \varepsilon\,\sin(t),$$
$$\Upsilon_2(t) = -\cos(10\,t) + \varepsilon\,\sin(t). \tag{12.8}$$

where $\varepsilon = 10^{-3}$. For this problem, we use $\omega = \max\Big(1, |\vartheta|\Big)$.

For $0 \le t \le 10000$, numerical solutions to the problem (12.7) have been established by utilizing the corresponding methods described in section 12.1.

The information in Figure 12 allows us to see the following:

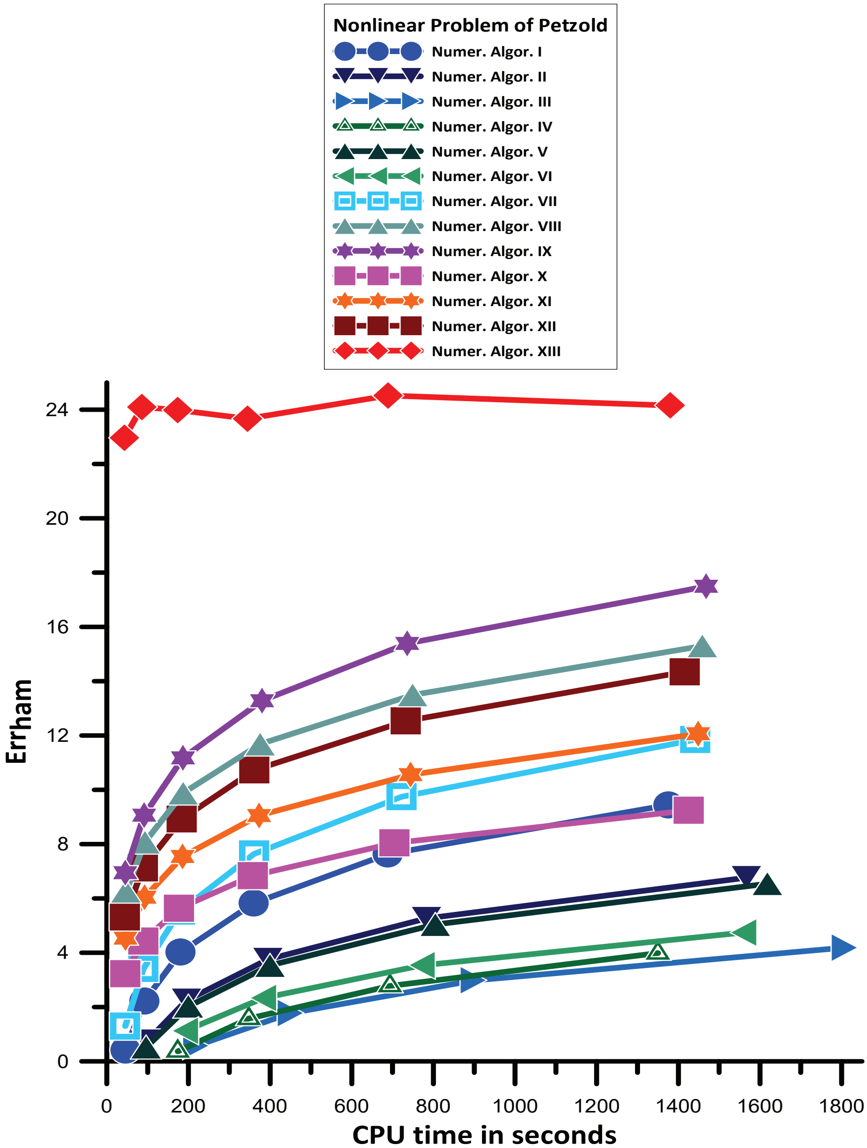


FIGURE 11. Numerical results for the nonlinear problem of [41]

- Numer. Algor. VII is more efficient than Numer. Algor. I.
- Numer. Algor. II gives mixed results. For big step sizes is more efficient than Numer. Algor. VII. For small step sizes is less efficient than Numer. Algor. VII.
- Numer. Algor. IV is more efficient than Numer. Algor. II.

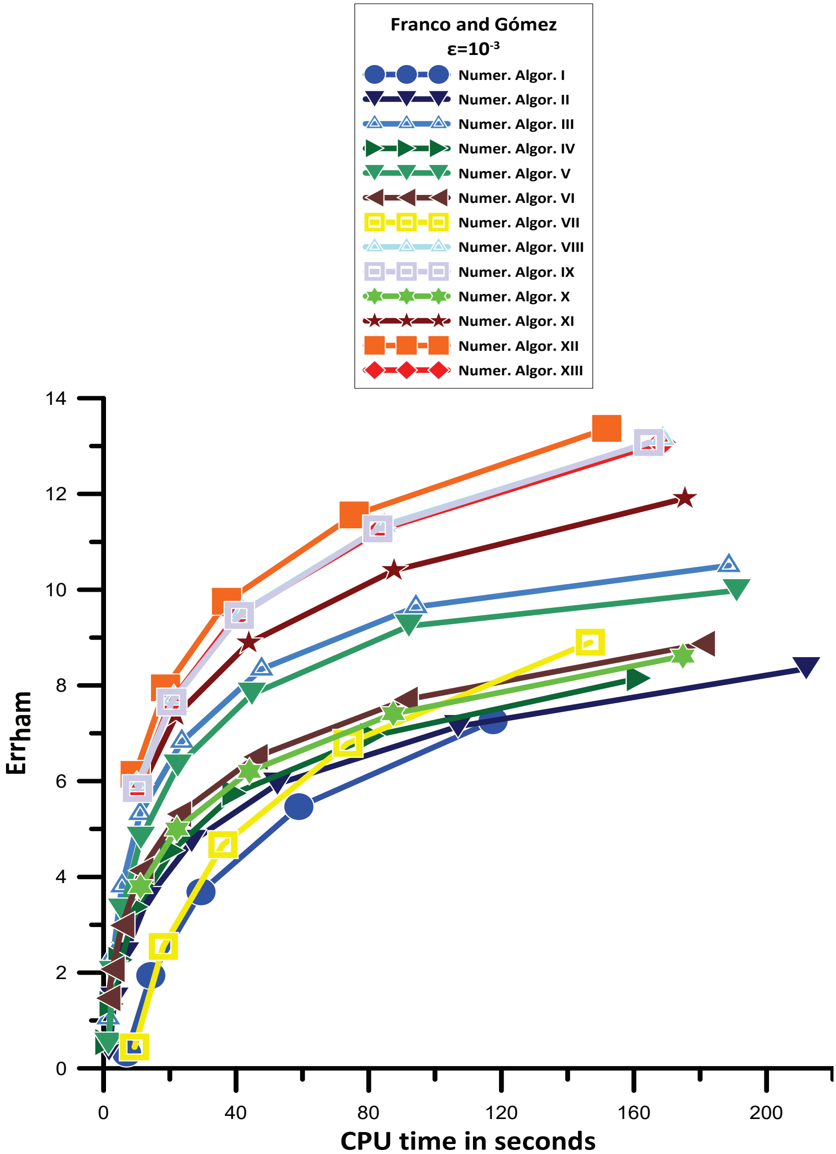


FIGURE 12. Numerical results for the Semi-linear Problem of Franco and Gómez [39]

- Numer. Algor. X is more efficient than Numer. Algor. IV.
- Numer. Algor. VI is more efficient than Numer. Algor. X.
- Numer. Algor. V is more efficient than Numer. Algor. VI.
- Numer. Algor. III is more efficient than Numer. Algor. V.

- Numer. Algor. XI is more efficient than Numer. Algor. III.
- Numer. Algor. XIII is more efficient than Numer. Algor. XI.
- Numer. Algor. VIII, Numer. Algor. IX, and Numer. Algor. XIII have the same, approximately, accuracy.
- Finally, Numer. Algor. XII, is the most efficient one.

12.5. A Nonlinear Orbital Problem [40]

The following nonlinear orbital problem was studied by Simos in [40] and is taken into account here:

$$\begin{aligned}
\Upsilon_1''(t) &= -\varphi^2\,\Upsilon_1(t) + \frac{2\,\Upsilon_1(t)\,\Upsilon_2(t) - \sin(2\,\varphi\,t)}{\left(\Upsilon_1(t)^2 + \Upsilon_2(t)^2\right)^{\frac{3}{2}}}, \quad \Upsilon_1(0) = 1, \quad \Upsilon_1'(0) = 0, \\
\Upsilon_2''(t) &= -\varphi^2\,\Upsilon_2(t) + \frac{\Upsilon_1(t)^2 - \Upsilon_2(t)^2 - \cos(2\,\varphi\,t)}{\left(\Upsilon_1(t)^2 + \Upsilon_2(t)^2\right)^{\frac{3}{2}}}, \quad \Upsilon_2(0) = 0, \quad \Upsilon_2'(0) = \varphi
\end{aligned} \tag{12.9}$$

The exact solution is

$$\Upsilon_1(t) = \cos(\varphi\,t), \quad \Upsilon_2(t) = \sin(\varphi\,t). \tag{12.10}$$

where $\varphi = 10$. For this problem, we use $\omega = 10$.

For $0 \leq t \leq 100000$, numerical solutions to the problem (12.9) have been established by utilizing the corresponding methods described in section 12.1.

The information in Figure 12 allows us to see the following:

- Numer. Algor. II is more efficient than Numer. Algor. I.
- Numer. Algor. VII gives mixed results. For big step sizes is more efficient than Numer. Algor. I and less efficient than Numer. Algor. II. For small step sizes is more efficient than Numer. Algor. I and Numer. Algor. II.
- Numer. Algor. IV gives mixed results. For big step sizes is more efficient than Numer. Algor. VII and and Numer. Algor. II. For small step sizes is more efficient than Numer. Algor. II and less efficient than Numer. Algor. VII.
- Numer. Algor. VI is more efficient than Numer. Algor. IV, Numer. Algor. VII, and Numer. Algor. II.
- Numer. Algor. V is more efficient than Numer. Algor. VI.
- Numer. Algor. III is more efficient than Numer. Algor. V.
- Numer. Algor. IX is more efficient than Numer. Algor. III.
- Numer. Algor. XI gives mixed results. For big step sizes is more efficient than Numer. Algor. IX. For small step sizes is less efficient than Numer. Algor. IX.
- Numer. Algor. VIII gives mixed results. For big step sizes is more efficient than Numer. Algor. XI. For small step sizes gives, approximately, the same accuracy with Numer. Algor. XI.

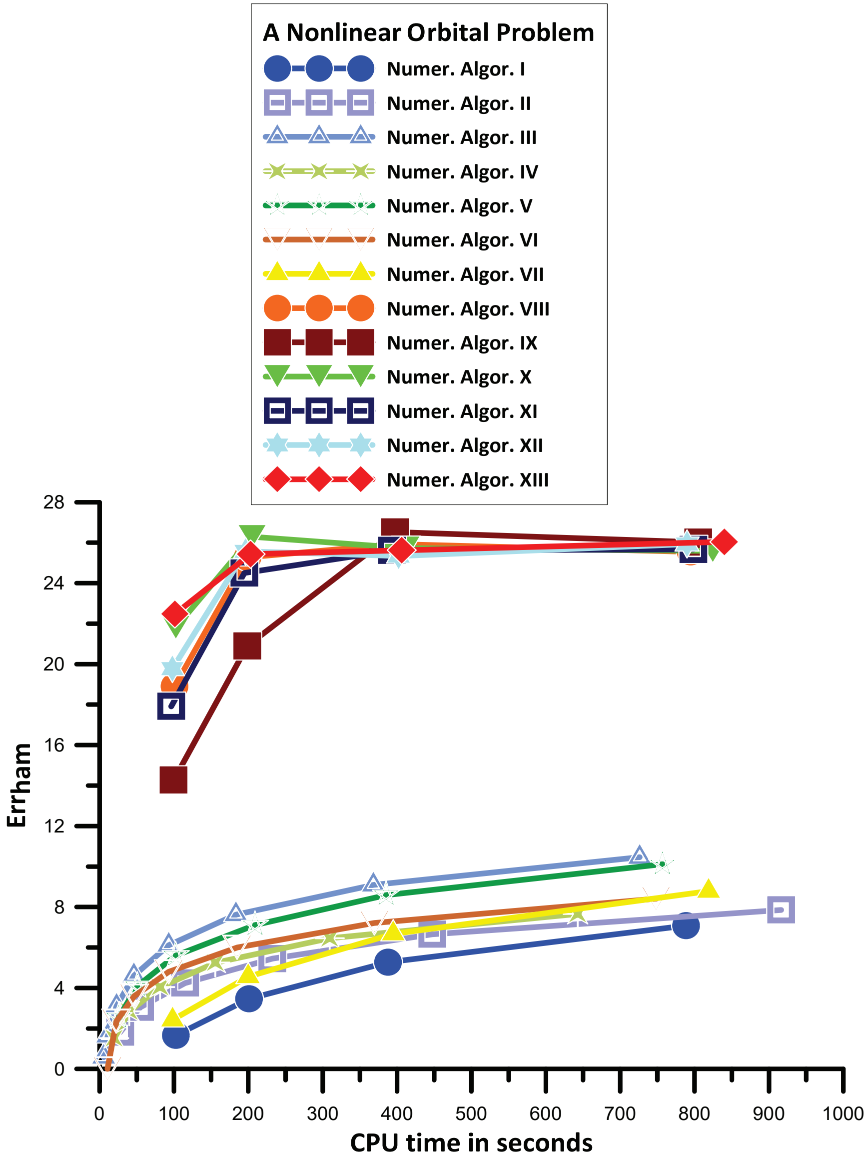


FIGURE 12. Numerical results for the Nonlinear Orbital problem of [40]

- Numer. Algor. XII gives mixed results. For big step sizes is more efficient than Numer. Algor. VIII. For small step sizes gives, approximately, the same accuracy with Numer. Algor. VIII.

- Numer. Algor. X gives mixed results. For big step sizes is more efficient than Numer. Algor. XII. For small step sizes gives, approximately, the same accuracy with Numer. Algor. XII.
- Numer. Algor. XIII, gives, approximately, the same accuracy with Numer. Algor. X.
- Finally, Numer. Algor. X, and Numer. Algor. XIII , are the most efficient ones.

12.6. Perturbed Two–Body Gravitational Problem - Case $\mu = 0.4$

We take into account the perturbed two-body Kepler's problem:

$$\Upsilon_1''(x) = -\frac{\Upsilon_1(x)}{\left(\Upsilon_1(x)^2 + \Upsilon_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu + 2\right)\frac{\Upsilon_1(x)}{\left(\Upsilon_1(x)^2 + \Upsilon_2(x)^2\right)^{\frac{5}{2}}},$$
$$\Upsilon_1(0) = 1, \quad \Upsilon_1'(0) = 0$$
$$\Upsilon_2''(x) = -\frac{\Upsilon_2(x)}{\left(\Upsilon_1(x)^2 + \Upsilon_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu + 2\right)\frac{\Upsilon_2(x)}{\left(\Upsilon_1(x)^2 + \Upsilon_2(x)^2\right)^{\frac{5}{2}}},$$
$$\Upsilon_2(0) = 0, \quad \Upsilon_2'(0) = 1 + \mu. \tag{12.11}$$

The exact solution is

$$\begin{aligned}\Upsilon_1(x) &= \cos(x + \mu\, x),\\ \Upsilon_2(x) &= \sin(x + \mu\, x).\end{aligned} \tag{12.12}$$

For this problem, we use $\omega = \frac{\sqrt{1+\mu\left(\mu+2\right)}}{\left(\Phi_1(x)^2+\Phi_2(x)^2\right)^{\frac{3}{4}}}$.

For $0 \leq t \leq 100000$, numerical solutions to the problem (12.11) have been established by utilizing the corresponding methods described in section 12.1.

The information in Figure 13 allows us to see the following:

- Numer. Algor. V is more efficient than Numer. Algor. VI.
- Numer. Algor. II, Numer. Algor. IV, and Numer. Algor. V have the same, approximately, accuracy.
- Numer. Algor. III is more efficient than Numer. Algor. V.
- Numer. Algor. VII is more efficient than Numer. Algor. III.
- Numer. Algor. I is more efficient than Numer. Algor. VII.
- Numer. Algor. IX is more efficient than Numer. Algor. I.
- Numer. Algor. VIII, Numer. Algor. IX, Numer. Algor. X, Numer. Algor. XI, Numer. Algor. XII, and Numer. Algor. XIII have the same, approximately, accuracy.
- Finally, Numer. Algor. VIII, Numer. Algor. IX, Numer. Algor. X, Numer. Algor. XI, Numer. Algor. XII, and Numer. Algor. XIII , are the most efficient ones.

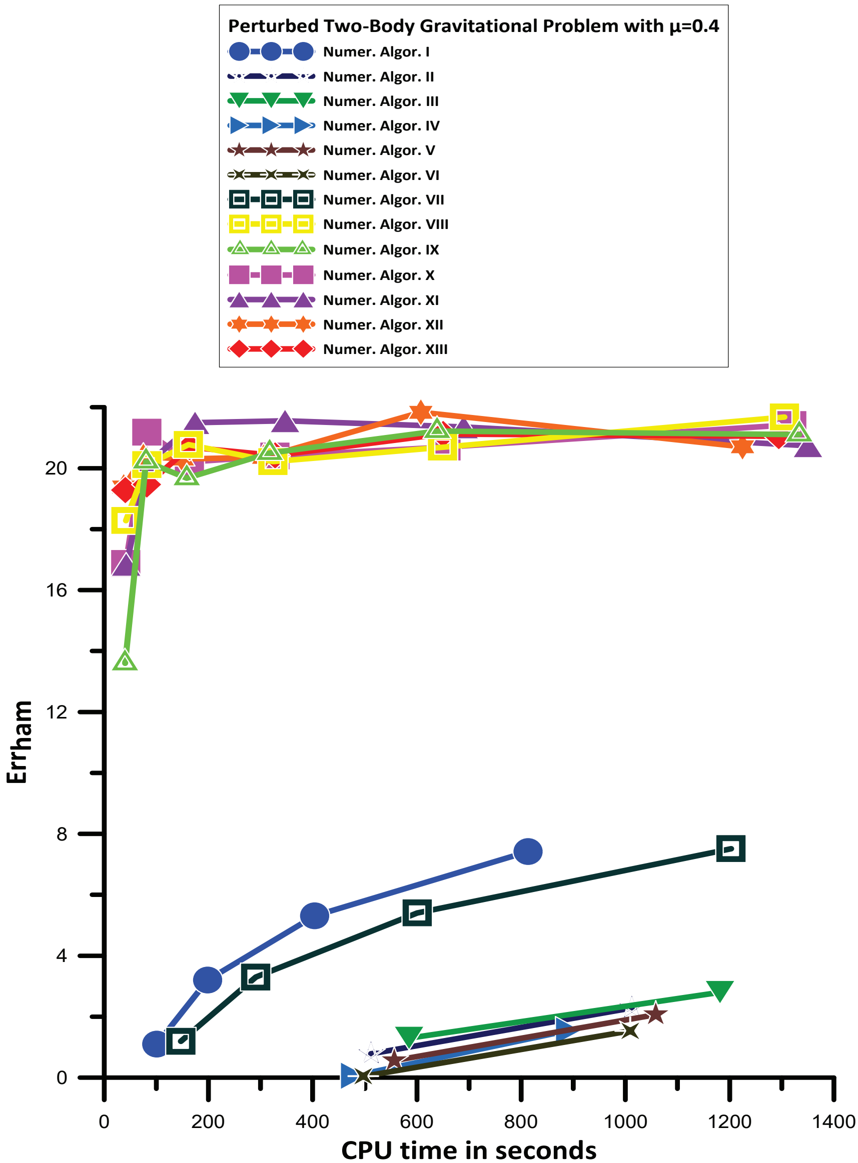


FIGURE 13. Numerical results for perturbed two–body gravitational problem (perturbed Kepler's problem) with $\mu = 0.4$

12.7. Conclusions from the Numerical Results

Considering all factors, the following approaches have proven to be the most effective (in the order of effectiveness):

- the method produced by the strategy of eliminating the phase–lag and amplification–factor together with the first and second derivatives of the phase–lag. This method developed in Section 9.
- the method produced by the strategy of eliminating the phase–lag and amplification–factor together with the first and second derivatives of the amplification–factor. This method developed in Section 10.
- the method produced by the strategy of eliminating the phase–lag and amplification–factor together with the first derivative of the phase–lag. This method developed in Section 6.
- the method produced by the strategy of eliminating the phase–lag and amplification–factor together with the first derivative of the amplification–factor. This method developed in Section 7.
- the method produced by the strategy of eliminating the phase–lag and amplification–factor. This method developed in Section 5.
- the method produced by the strategy of eliminating the phase–lag and amplification–factor together with the first derivative of the phase–lag and the first derivative of the amplification–factor. This method developed in Section 8.

The effectiveness of frequency–dependent algorithms, such the ones that were recently suggested, depends on finding the best value of the parameter v. This decision is defined explicitly in the problem model for many problems. The parameter v can be found using methods suggested in the literature for cases where this is not easy to do (refer to [44] and [45]).

12.8. High–order Ordinary Differential Equations and Partial Differential Equations

Keep in mind that there are already known methods to reduce systems of high-order ordinary differential equations to first-order differential equations before attempting to solve them using the newly presented methods. For further details, see [42]. Some examples of these methods involve changing the values of variables, introducing new variables, or rewriting the system with new variables for each derivative.

Prior to mentioning the newly introduced techniques, it is worth mentioning that there are already known ways to reduce a system of partial differential equations to a system of first-order differential equations, like the characteristics method (see [43]).

13. Conclusions

Using the sixth algebraic order Adams–Bashforth method ($AB6AOM$) as a foundation, we set out to develop effective algorithms. We then detailed many ways to efficiently create methods that eliminate phase–lag, amplification–factor, and their derivatives in light of the foregoing. We specifically set up processes for:

- elimination of the amplification–factor with minimization of the phase–lag (Section 4)
- elimination of the phase–lag and the amplification–factor (Section 5)
- elimination of the phase–lag and the amplification–factor with elimination of the 1st derivative of the phase–lag (Section 6)
- elimination of the phase–lag and the amplification–factor with elimination of the 1st derivative of the amplification–factor (Section 7)
- elimination of the phase–lag and the amplification–factor with elimination of the 1st derivative of the phase–lag and the 1st derivative of the amplification–factor (Section 8)
- elimination of the phase–lag and the amplification–factor with elimination of the 1st and 2nd derivatives of the phase–lag (Section 9)
- elimination of the phase–lag and the amplification–factor with elimination of the 1st and 2nd derivatives of the amplification–factor (Section 10)

For the above obtained methods we presented their stability analysis (Section 11).

The effectiveness of the aforementioned techniques was tested on multiple problems with oscillating solutions.

All calculations adhered to the IEEE Standard 754 and were executed on a personal computer featuring an x86 − −64 compatible architecture and utilizing a quadruple precision arithmetic data type consisting of 64 bits.

Appendix A - $Formula_j$, $j = 5\,(1)\,8$

$$
\begin{aligned}
Formula_5 &= \frac{(720\,A_3(v)+3649)\,(-21600\,A_3(v)-10470)}{720\,A_3(v)+4969} && (13.1)\\
Formula_6 &= 2916\,A_3(v)+\frac{295569}{20}+\frac{720\,A_3(v)+3649}{720\,A_3(v)+4969}\left(70200\,A_3(v)+\frac{299055}{2}\right)\\
&- \frac{1}{2}\,\frac{Formula_{61}}{\left(720\,A_3(v)+4969\right)^2} && (13.2)\\
Formula_{61} &= \left(6739200\,A_3(v)^2+47400480\,A_3(v)+67130653\right)\\
&\quad \left(-21600\,A_3(v)-10470\right)
\end{aligned}
$$

$$
\begin{aligned}
Formula_7 &= -\frac{4374\,A_3\left(v\right)}{7}-\frac{1882891}{840}\\
&+ \frac{720\,A_3\left(v\right)+3649}{720\,A_3\left(v\right)+4969}\Big(-61860\,A_3\left(v\right)-\frac{4029401}{28}\Big)\\
&- \frac{1}{2}\,\frac{6739200\,A_3\left(v\right)^2+47400480\,A_3\left(v\right)+67130653}{\left(720\,A_3\left(v\right)+4969\right)^2}\\
&\Big(70200\,A_3\left(v\right)+\frac{299055}{2}\Big)+\frac{1}{40}\,\frac{Formula_{71}}{\left(720\,A_3\left(v\right)+4969\right)^3} \qquad (13.3)\\
Formula_{71} &= \Big(737911296000\,A_3\left(v\right)^3+7258895481600\,A_3\left(v\right)^2\\
&+ 22343778267120\,A_3\left(v\right)+22850252915153\Big)\\
&\Big(-21600\,A_3\left(v\right)-10470\Big)\\
Formula_8 &= \frac{2187\,A_3\left(v\right)}{28}-\frac{9357949}{20160}+\frac{720\,A_3\left(v\right)+3649}{720\,A_3\left(v\right)+4969}\\
&\Big(\frac{389985\,A_3\left(v\right)}{14}+\frac{151134421}{2016}\Big)\\
&- \frac{1}{2}\,\frac{6739200\,A_3\left(v\right)^2+47400480\,A_3\left(v\right)+67130653}{\left(720\,A_3\left(v\right)+4969\right)^2}\\
&\Big(-61860\,A_3\left(v\right)-\frac{4029401}{28}\Big)\\
&+ \frac{1}{40}\,\frac{Formula_{81}}{\Big(720\,A_3\left(v\right)+4969\Big)^3}\Big(70200\,A_3\left(v\right)+\frac{299055}{2}\Big)\\
&- \frac{Formula_{82}}{5040\,\left(720\,A_3\left(v\right)+4969\right)^4} \qquad (13.4)\\
Formula_{81} &= 737911296000\,A_3\left(v\right)^3+7258895481600\,A_3\left(v\right)^2\\
&+ 22343778267120\,A_3\left(v\right)+22850252915153\\
Formula_{82} &= \Big(497370308567040000\,A_3\left(v\right)^4+6169055214154752000\,A_3\left(v\right)^3\\
&+ 27521512334145561600\,A_3\left(v\right)^2+53091622818281299680\,A_3\left(v\right)\\
&+ 37095094510251921349\Big)\Big(-21600\,A_3\left(v\right)-10470\Big)
\end{aligned}
$$

Appendix B - $Formula_q$, $q = 17\,(1)\,19$

$$
\begin{aligned}
Formula_{17} &= -34560\,(\cos(v))^6\,v + 11520\,(\cos(v))^5\,\sin(v) \\
&+ 16796\,(\cos(v))^3\,\sin(v)\,v^2 - 23040\,(\cos(v))^5\,v \\
&+ 5760\,(\cos(v))^4\,\sin(v) + 13919\,(\cos(v))^2\,\sin(v)\,v^2 \\
&+ 48960\,(\cos(v))^4\,v - 11520\,(\cos(v))^3\,\sin(v) \\
&- 6601\,\sin(v)\,\cos(v)\,v^2 + 27360\,(\cos(v))^3\,v \\
&- 4320\,(\cos(v))^2\,\sin(v) - 3724\,v^2\,\sin(v) \\
&- 17280\,(\cos(v))^2\,v + 2160\,\sin(v)\,\cos(v) \\
&- 6480\,v\,\cos(v) + 360\,\sin(v) + 1080\,v, \qquad (13.5) \\
Formula_{18} &= -11520\,(\cos(v))^8\,v + 11520\,(\cos(v))^7\,\sin(v) \\
&- 12996\,(\cos(v))^5\,\sin(v)\,v^2 - 11520\,(\cos(v))^7\,v \\
&+ 5760\,(\cos(v))^6\,\sin(v) - 21627\,(\cos(v))^4\,\sin(v)\,v^2 \\
&+ 14400\,(\cos(v))^6\,v - 14400\,(\cos(v))^5\,\sin(v) \\
&- 5382\,(\cos(v))^3\,\sin(v)\,v^2 + 14400\,(\cos(v))^5\,v \\
&- 5760\,(\cos(v))^4\,\sin(v) + 6126\,(\cos(v))^2\,\sin(v)\,v^2 \\
&- 5040\,(\cos(v))^4\,v + 5040\,(\cos(v))^3\,\sin(v) \\
&+ 3808\,\sin(v)\,\cos(v)\,v^2 - 5040\,(\cos(v))^3\,v \\
&+ 1440\,(\cos(v))^2\,\sin(v) + 931\,v^2\,\sin(v) \\
&+ 1080\,(\cos(v))^2\,v - 540\,\sin(v)\,\cos(v) \\
&+ 540\,v\,\cos(v) - 90\,\sin(v) - 270\,v, \qquad (13.6) \\
Formula_{19} &= 7600\,(\cos(v))^5\,\sin(v)\,v^2 - 15416\,(\cos(v))^4\,\sin(v)\,v^2 \\
&- 11520\,(\cos(v))^6\,v + 11520\,(\cos(v))^5\,\sin(v) \\
&- 42662\,(\cos(v))^3\,\sin(v)\,v^2 - 11520\,(\cos(v))^5\,v \\
&+ 5760\,(\cos(v))^4\,\sin(v) - 13892\,(\cos(v))^2\,\sin(v)\,v^2 \\
&+ 11520\,(\cos(v))^4\,v - 11520\,(\cos(v))^3\,\sin(v) \\
&+ 14627\,\sin(v)\,\cos(v)\,v^2 + 11520\,(\cos(v))^3\,v \\
&- 4320\,(\cos(v))^2\,\sin(v) + 8873\,v^2\,\sin(v) \\
&- 720\,(\cos(v))^2\,v + 2160\,\sin(v)\,\cos(v) \\
&- 2160\,v\,\cos(v) + 360\,\sin(v) - 1080\,v. \qquad (13.7)
\end{aligned}
$$

Appendix C - $Formula_p$, $p = 20,\ 21$

$$
\begin{aligned}
Formula_{20} &= -228000 - 38016000\, v\, A_0(v)\, \sin(5v) - 2073600\, v\, A_2(v)\, \sin(3v) \\
&- 1382400\, v\, A_3(v)\, \sin(2v) + 15212160\, v\, \sin(4v) \\
&- 4147200\, \cos(6v) + 3456000\, \cos(5v) \\
&- 34524000\, \sin(v)\, v^3\, A_0(v) - 2764800\, \cos(2v)\, v^2\, A_3(v)^2 \\
&- 3803040\, \cos(4v) + 1380960\, \cos(v) + 691200\, A_2(v)\, \cos(3v) \\
&+ 691200\, A_3(v)\, \cos(2v) - 37324800\, \cos(6v)\, v^2\, A_2(v) \\
&+ 691200\, \cos(5v)\, A_0(v) - 19616075\, v^2 \\
&- 103680000\, \cos(6v)\, v^2\, A_0(v) + 59467680\, \cos(2v)\, v^2\, A_3(v) \\
&+ 145867680\, \cos(5v)\, v^2\, A_0(v) + 178403040\, \sin(3v)\, v^3\, A_2(v) \\
&+ 380304000\, \sin(4v)\, v^3\, A_0(v) + 60848640\, \sin(4v)\, v^3\, A_3(v) \\
&- 86400000\, \sin(5v)\, v^3\, A_0(v)^2 + 34560000\, \sin(6v)\, v\, A_0(v) \\
&- 6220800\, \cos(3v)\, v^2\, A_2(v)^2 + 15212160\, \cos(4v)\, v^2\, A_3(v) \\
&+ 5529600\, \sin(6v)\, v\, A_3(v) + 13824000\, \cos(5v)\, v^2\, A_3(v) \\
&+ 136909440\, \sin(4v)\, v^3\, A_2(v) - 5529600\, \sin(5v)\, v\, A_3(v) \\
&+ 95076000\, \cos(4v)\, v^2\, A_0(v) + 34227360\, \cos(4v)\, v^2\, A_2(v) \\
&+ 5700000\, v^2\, A_0(v) - 5523840\, \sin(v)\, v^3\, A_3(v) \\
&+ 297338400\, \sin(5v)\, v^3\, A_0(v) - 5523840\, \cos(v)\, v^2\, A_3(v) \\
&- 16588800\, \cos(6v)\, v^2\, A_3(v) - 18662400\, \sin(3v)\, v^3\, A_2(v)^2 \\
&- 12441600\, \sin(5v)\, v\, A_2(v) + 118935360\, \sin(2v)\, v^3\, A_3(v) \\
&- 1308784524\, v^3 \sin(4v) - 327196131\, v^2 \cos(4v) \\
&- 297338400\, v^2 \cos(5v) - 118935360\, v\, \sin(6v) \\
&+ 356806080\, v^2 \cos(6v) - 12428640\, \cos(v)\, v^2\, A_2(v) \\
&+ 118811469\, v^3 \sin(v) + 2052000\, v^2\, A_2(v) \\
&+ 118935360\, v\, \sin(5v) - 1380960\, v\, \sin(v) \\
&+ 118811469\, v^2 \cos(v) - 34524000\, \cos(v)\, v^2\, A_0(v) \\
&+ 59467680\, \cos(3v)\, v^2\, A_2(v) - 5529600\, \sin(2v)\, v^3\, A_3(v)^2 \\
&+ 912000\, v^2\, A_3(v) + 31104000\, \cos(5v)\, v^2\, A_2(v) \\
&- 12428640\, \sin(v)\, v^3\, A_2(v) + 12441600\, \sin(6v)\, v\, A_2(v) \\
&- 17280000\, \cos(2v)\, v^2\, A_0(v)\, A_3(v) - 6220800\, \cos(2v)\, v^2\, A_2(v)\, A_3(v) \\
&- 2764800\, \cos(5v)\, v^2\, A_0(v)\, A_3(v) - 17280000\, \cos(3v)\, v^2\, A_0(v)\, A_2(v) \\
&- 2764800\, \cos(3v)\, v^2\, A_2(v)\, A_3(v) - 8294400\, \sin(3v)\, v^3\, A_2(v)\, A_3(v) \\
&- 13824000\, \sin(5v)\, v^3\, A_0(v)\, A_3(v) - 51840000\, \sin(3v)\, v^3\, A_0(v)\, A_2(v) \\
&- 34560000\, \sin(2v)\, v^3\, A_0(v)\, A_3(v) - 12441600\, \sin(2v)\, v^3\, A_2(v)\, A_3(v) \\
&- 6220800\, \cos(5v)\, v^2\, A_0(v)\, A_2(v) - 31104000\, \sin(5v)\, v^3\, A_0(v)\, A_2(v) \\
&- 17280000\, \cos(5v)\, v^2\, A_0(v)^2, \qquad (13.8) \\
Formula_{21} &= 432000000\, v^4\, A_0(v)^2 + 311040000\, v^4\, A_0(v)\, A_2(v) \\
&+ 138240000\, v^4\, A_0(v)\, A_3(v) + 55987200\, v^4\, A_2(v)^2 \\
&+ 49766400\, v^4\, A_2(v)\, A_3(v) + 11059200\, v^4\, A_3(v)^2
\end{aligned}
$$

Appendix D - $Formula_r$, $r = 22\,(1)\,24$

$$
\begin{aligned}
Formula_{22} &= -69120\,(\cos(v))^6\,v + 23040\sin(v)\,(\cos(v))^5 \\
&+ 33592\sin(v)\,(\cos(v))^3\,v^2 + 23040\,(\cos(v))^5\,v \\
&- 11520\sin(v)\,(\cos(v))^4 - 5754\sin(v)\,(\cos(v))^2\,v^2 \\
&+ 109440\,(\cos(v))^4\,v - 23040\sin(v)\,(\cos(v))^3 \\
&- 24244\,v^2\sin(v)\cos(v) - 31680\,(\cos(v))^3\,v \\
&+ 8640\sin(v)\,(\cos(v))^2 + 2877\,v^2\sin(v) \\
&- 43200\,(\cos(v))^2\,v + 4320\sin(v)\cos(v) \\
&+ 8640\,v\cos(v) - 720\sin(v) + 2160\,v, \qquad (13.10) \\
Formula_{23} &= -15360\,(\cos(v))^8\,v + 15360\sin(v)\,(\cos(v))^7 \\
&- 17328\sin(v)\,(\cos(v))^5\,v^2 - 7680\sin(v)\,(\cos(v))^6 \\
&- 11508\sin(v)\,(\cos(v))^4\,v^2 + 26880\,(\cos(v))^6\,v \\
&- 19200\sin(v)\,(\cos(v))^5 + 12996\sin(v)\,(\cos(v))^3\,v^2 \\
&+ 7680\sin(v)\,(\cos(v))^4 + 9590\sin(v)\,(\cos(v))^2\,v^2 \\
&- 14400\,(\cos(v))^4\,v + 6720\sin(v)\,(\cos(v))^3 \\
&+ 1558\,v^2\sin(v)\cos(v) - 1920\sin(v)\,(\cos(v))^2 \\
&- 959\,v^2\sin(v) + 3360\,(\cos(v))^2\,v \\
&- 720\sin(v)\cos(v) - 720\,v\cos(v) \\
&+ 120\sin(v) - 120\,v, \qquad (13.11) \\
Formula_{24} &= 15200\sin(v)\,(\cos(v))^6\,v^2 - 46032\sin(v)\,(\cos(v))^5\,v^2 \\
&- 23040\,(\cos(v))^7\,v + 23040\sin(v)\,(\cos(v))^6 \\
&- 46892\sin(v)\,(\cos(v))^4\,v^2 - 11520\sin(v)\,(\cos(v))^5 \\
&+ 34524\sin(v)\,(\cos(v))^3\,v^2 + 34560\,(\cos(v))^5\,v \\
&- 23040\sin(v)\,(\cos(v))^4 + 35492\sin(v)\,(\cos(v))^2\,v^2 \\
&+ 8640\sin(v)\,(\cos(v))^3 - 10080\,(\cos(v))^3\,v \\
&+ 4320\sin(v)\,(\cos(v))^2 - 4199\,v^2\sin(v) \\
&- 2880\,(\cos(v))^2\,v - 720\sin(v)\cos(v) + 720\,v. \qquad (13.12)
\end{aligned}
$$

Appendix E - $Formula_s,\ s = 25\,(1)\,29$

$$\begin{aligned}
Formula_{25} &= -480\,v\,A_0\,(v)\,\sin(5\,v) - 480\,v\,A_2\,(v)\,\sin(3\,v) \\
&- 480\,v\,A_3\,(v)\,\sin(2\,v) - 480\,v\,A_4\,(v)\,\sin(v) \\
&+ 2641\,v\sin(4\,v) - 480\,\cos(6\,v) + 480\,\cos(5\,v)\,, && (13.13) \\
Formula_{26} &= 1440\,v\,A_0\,(v)\,\cos(5\,v) + 1440\,v\,A_2\,(v)\,\cos(3\,v) \\
&+ 1440\,v\,A_3\,(v)\,\cos(2\,v) + 1440\,v\,A_4\,(v)\,\cos(v) \\
&- 7923\,v\cos(4\,v) + 1440\,\sin(5\,v) \\
&- 1440\,\sin(6\,v) - 475\,v, && (13.14) \\
Formula_{27} &= -480\,A_0\,(v)\,\sin(5\,v) - 2400\,v\,A_0\,(v)\,\cos(5\,v) \\
&- 480\,A_2\,(v)\,\sin(3\,v) - 1440\,v\,A_2\,(v)\,\cos(3\,v) \\
&- 480\,A_3\,(v)\,\sin(2\,v) - 960\,v\,A_3\,(v)\,\cos(2\,v) \\
&- 480\,\sin(v)\,A_4 - 480\,v\,A_4\,(v)\,\cos(v) \\
&+ 2641\,\sin(4\,v) + 10564\,v\cos(4\,v) \\
&+ 2880\,\sin(6\,v) - 2400\,\sin(5\,v)\,, && (13.15) \\
Formula_{28} &= -14250 - 1944000\,\sin(5\,v)\,v^3\,A_0\,(v)\,A_2\,(v) \\
&- 864000\,\sin(5\,v)\,v^3\,A_0\,(v)\,A_3\,(v) \\
&- 216000\,\sin(5\,v)\,v^3\,A_0\,(v)\,A_4\,(v) - 3240000\,\sin(3\,v)\,v^3\,A_0\,(v)\,A_2\,(v) \\
&- 518400\,\sin(3\,v)\,v^3\,A_2\,(v)\,A_3\,(v) - 129600\,\sin(3\,v)\,v^3\,A_2\,(v)\,A_4\,(v) \\
&- 2160000\,\sin(2\,v)\,v^3\,A_0\,(v)\,A_3\,(v) - 777600\,\sin(2\,v)\,v^3\,A_2\,(v)\,A_3\,(v) \\
&- 86400\,\sin(2\,v)\,v^3\,A_3\,(v)\,A_4\,(v) - 2376000\,v\,A_0\,(v)\,\sin(5\,v) \\
&- 129600\,v\,A_2\,(v)\,\sin(3\,v) - 86400\,v\,A_3\,(v)\,\sin(2\,v) \\
&- 43200\,v\,A_4\,(v)\,\sin(v) + 950760\,v\sin(4\,v) \\
&- 259200\,\cos(6\,v) + 216000\,\cos(5\,v) \\
&- 172800\,\cos(v)\,v^2\,A_3\,(v)\,A_4\,(v) - 388800\,\cos(v)\,v^2\,A_2\,(v)\,A_4\,(v) \\
&- 1080000\,\cos(v)\,v^2\,A_0\,(v)\,A_4\,(v) - 43200\,\cos(2\,v)\,v^2\,A_3\,(v)\,A_4\,(v) \\
&- 388800\,\cos(2\,v)\,v^2\,A_2\,(v)\,A_3\,(v) - 1080000\,\cos(2\,v)\,v^2\,A_0\,(v)\,A_3\,(v) \\
&- 43200\,\cos(3\,v)\,v^2\,A_2\,(v)\,A_4\,(v) - 172800\,\cos(3\,v)\,v^2\,A_2\,(v)\,A_3\,(v) \\
&- 1080000\,\cos(3\,v)\,v^2\,A_0\,(v)\,A_2\,(v) - 172800\,\cos(5\,v)\,v^2\,A_0\,(v)\,A_3\,(v) \\
&- 43200\,\cos(5\,v)\,v^2\,A_0\,(v)\,A_4\,(v) - 388800\,\cos(5\,v)\,v^2\,A_0\,(v)\,A_2\,(v) \\
&- 172800\,\sin(v)\,v^3\,A_3\,(v)\,A_4\,(v) - 1080000\,\sin(v)\,v^3\,A_0\,(v)\,A_4\,(v) \\
&- 388800\,\sin(v)\,v^3\,A_2\,(v)\,A_4\,(v) - 237690\,\cos(4\,v) \\
&+ 43200\,A_2\,(v)\,\cos(3\,v) + 43200\,A_3\,(v)\,\cos(2\,v) \\
&+ 43200\,\cos(v)\,A_4 + 43200\,\cos(5\,v)\,A_0 \\
&- 259200\,\cos(6\,v)\,v^2\,A_4\,(v) - 1254475\,v^2
\end{aligned}$$

$$
\begin{aligned}
& - && 1080000 \cos(5v) v^2 A_0(v)^2 + 216000 \cos(5v) v^2 A_4(v) \\
& + && 1944000 \cos(5v) v^2 A_2(v) - 43200 \sin(v) v^3 A_4(v)^2 \\
& + && 11409120 \sin(3v) v^3 A_2(v) + 9203040 \cos(5v) v^2 A_0(v) \\
& + && 23769000 \sin(4v) v^3 A_0(v) - 6480000 \cos(6v) v^2 A_0(v) \\
& - && 388800 \cos(3v) v^2 A_2(v)^2 + 3803040 \cos(2v) v^2 A_3(v) \\
& + && 950760 \cos(4v) v^2 A_3(v) + 3803040 \cos(3v) v^2 A_2(v) \\
& + && 950760 \sin(4v) v^3 A_4(v) + 5942250 \cos(4v) v^2 A_0(v) \\
& + && 2139210 \cos(4v) v^2 A_2(v) + 3803040 \sin(v) v^3 A_4(v) \\
& + && 3803040 \sin(4v) v^3 A_3(v) - 345600 \sin(5v) v A_3(v) \\
& - && 777600 \sin(5v) v A_2(v) - 345600 \sin(2v) v^3 A_3(v)^2 \\
& + && 86400 \sin(6v) v A_4(v) - 5400000 \sin(5v) v^3 A_0(v)^2 \\
& + && 19015200 \sin(5v) v^3 A_0(v) - 43200 \cos(v) v^2 A_4(v)^2 \\
& - && 1036800 \cos(6v) v^2 A_3(v) + 7606080 \sin(2v) v^3 A_3(v) \\
& - && 172800 \cos(2v) v^2 A_3(v)^2 + 864000 \cos(5v) v^2 A_3(v) \\
& - && 86400 \sin(5v) v A_4(v) + 2160000 \sin(6v) v A_0(v) \\
& + && 14250 v^2 A_4(v) + 57000 v^2 A_3(v) + 128250 v^2 A_2(v) \\
& + && 356250 v^2 A_0(v) + 3803040 \cos(v) v^2 A_4(v) \\
& + && 8556840 \sin(4v) v^3 A_2(v) - 2332800 \cos(6v) v^2 A_2(v) \\
& - && 83698572 v^3 \sin(4v) - 19015200 v^2 \cos(5v) \\
& - && 7606080 v \sin(6v) + 22818240 v^2 \cos(6v) \\
& - && 20924643 v^2 \cos(4v) + 7606080 v \sin(5v) \\
& - && 1166400 \sin(3v) v^3 A_2(v)^2 + 237690 \cos(4v) v^2 A_4(v) \\
& + && 345600 \sin(6v) v A_3(v) + 777600 \sin(6v) v A_2(v), \qquad (13.16)
\end{aligned}
$$

$$
\begin{aligned}
Formula_{29} & = && 27000000 v^4 A_0(v)^2 + 19440000 v^4 A_0(v) A_2(v) \\
& + && 8640000 v^4 A_0(v) A_3(v) + 2160000 v^4 A_0(v) A_4(v) \\
& + && 3499200 v^4 A_2(v)^2 + 3110400 v^4 A_2(v) A_3(v) \\
& + && 777600 v^4 A_2(v) A_4(v) + 691200 v^4 A_3(v)^2 \\
& + && 345600 v^4 A_3(v) A_4(v) + 43200 v^4 A_4(v)^2 \\
& - && 190152000 v^4 A_0(v) - 68454720 v^4 A_2(v) \\
& - && 30424320 v^4 A_3(v) - 7606080 v^4 A_4(v) \\
& + && 334794288 v^4 + 2160000 v^2 A_0(v) \\
& + && 777600 v^2 A_2(v) + 345600 v^2 A_3(v) \\
& + && 86400 v^2 A_4(v) - 7606080 v^2 + 43200. \qquad (13.17)
\end{aligned}
$$

Appendix F - $Formula_t$, $t = 30\,(1)\,33$

$$\begin{aligned}
Formula_{30} &= -2160\,(\cos(v))^4\,v + 720\,(\cos(v))^3\,\sin(v) \\
&+ 931\,v^2\,\cos(v)\,\sin(v) - 1440\,(\cos(v))^3\,v \\
&+ 360\,(\cos(v))^2\,\sin(v) + 931\,\sin(v)\,v^2 \\
&+ 2160\,(\cos(v))^2\,v - 360\,\sin(v)\,\cos(v) \\
&+ 1080\,v\,\cos(v) - 90\,\sin(v) - 270\,v, \qquad (13.18) \\
Formula_{31} &= -1440\,(\cos(v))^6\,v + 1440\,(\cos(v))^5\,\sin(v) \\
&- 2337\,(\cos(v))^3\,\sin(v)\,v^2 - 1440\,(\cos(v))^5\,v \\
&+ 720\,(\cos(v))^4\,\sin(v) - 2337\,(\cos(v))^2\,\sin(v)\,v^2 \\
&+ 720\,(\cos(v))^4\,v - 720\,(\cos(v))^3\,\sin(v) \\
&- 931\,v^2\,\cos(v)\,\sin(v) + 720\,(\cos(v))^3\,v \\
&- 180\,(\cos(v))^2\,\sin(v) - 931\,\sin(v)\,v^2 \\
&- 180\,(\cos(v))^2\,v + 180\,\sin(v)\,\cos(v) \\
&- 180\,v\,\cos(v) + 90\,\sin(v) + 270\,v, \qquad (13.19) \\
Formula_{32} &= 950\,(\cos(v))^3\,\sin(v)\,v^2 + 2880\,(\cos(v))^4\,v \\
&+ 950\,(\cos(v))^2\,\sin(v)\,v^2 - 2880\,(\cos(v))^3\,\sin(v) \\
&+ 2880\,(\cos(v))^3\,v + 4199\,v^2\,\cos(v)\,\sin(v) \\
&- 1440\,(\cos(v))^2\,\sin(v) - 720\,(\cos(v))^2\,v \\
&+ 4199\,\sin(v)\,v^2 + 720\,\sin(v)\,\cos(v) \\
&- 720\,v\,\cos(v) - 720\,v, \qquad (13.20) \\
Formula_{33} &= 950\,(\cos(v))^3\,\sin(v)\,v^2 + 720\,(\cos(v))^4\,v \\
&+ 950\,(\cos(v))^2\,\sin(v)\,v^2 - 720\,(\cos(v))^3\,\sin(v) \\
&+ 720\,(\cos(v))^3\,v + 931\,v^2\,\cos(v)\,\sin(v) \\
&- 360\,(\cos(v))^2\,\sin(v) + 931\,\sin(v)\,v^2 \\
&- 90\,\sin(v) - 270\,v. \qquad (13.21)
\end{aligned}$$

Appendix G - $Formula_w$, $w = 35\,(1)\,40$

$$\begin{aligned}
Formula_{35} &= 1440\,v\,\sin(4\,v) - 2880\,\cos(6\,v) \\
&+ 2880\,\cos(5\,v) + 1440\,\cos(2\,v) \\
&- 1440\,\cos(v) - 1440\,v^2 \\
&- 8640\,v^2\,\cos(10\,v) + 5040\,v\,\sin(10\,v) \\
&+ 7923\,v^3\,\sin(8\,v) - 8398\,v^3\,\sin(6\,v) \\
&- 45638\,v^3\,\sin(4\,v) + 84778\,v^3\,\sin(2\,v) \\
&+ 1440\,\cos(10\,v) - 1440\,\cos(9\,v) \\
&- 1440\,v\,\sin(8\,v) + 4320\,v^2\,\cos(9\,v) \\
&- 2880\,v^2\,\cos(7\,v) - 21600\,v^2\,\cos(5\,v) \\
&+ 18720\,v^2\,\cos(3\,v) - 17280\,v\,\sin(6\,v) \\
&+ 12240\,v\,\sin(2\,v) - 3600\,v\,\sin(9\,v) \\
&+ 1440\,v\,\sin(7\,v) + 4320\,v^2\,\cos(8\,v) \\
&+ 37440\,v^2\,\cos(6\,v) - 20160\,v^2\,\cos(4\,v) \\
&- 46080\,v^2\,\cos(2\,v) + 14400\,v\,\sin(5\,v) \\
&- 1440\,v\,\sin(3\,v) - 10800\,v\,\sin(v) \\
&+ 36000\,v^2\,\cos(v)\,, \qquad (13.22) \\
Formula_{36} &= -1440 + 9360\,v\,\sin(4\,v) + 1920\,\cos(6\,v) \\
&- 1920\,\cos(5\,v) + 1440\,\cos(4\,v) \\
&- 1440\,\cos(3\,v) - 960\,\cos(2\,v) \\
&+ 2400\,\cos(v) + 56160\,v^2 - 1440\,\cos(12\,v) \\
&+ 7923\,v^3\,\sin(10\,v) + 2880\,v^2\,\cos(12\,v) \\
&- 3600\,v\,\sin(12\,v) + 1440\,\cos(11\,v) \\
&+ 2160\,v\,\sin(11\,v) + 960\,v^2\,\cos(10\,v) \\
&- 1440\,v\,\sin(10\,v) + 950\,v^3\,\sin(8\,v) \\
&- 34067\,v^3\,\sin(6\,v) - 31692\,v^3\,\sin(4\,v) \\
&+ 122170\,v^3\,\sin(2\,v) - 960\,\cos(10\,v) \\
&+ 960\,\cos(9\,v) + 7920\,v\,\sin(8\,v) \\
&- 5760\,v^2\,\cos(3\,v) + 11520\,v\,\sin(6\,v) \\
&- 10080\,v\,\sin(2\,v) + 480\,v\,\sin(9\,v) \\
&- 6480\,v\,\sin(7\,v) - 7200\,v^2\,\cos(8\,v) \\
&- 10560\,v^2\,\cos(6\,v) - 5760\,v^2\,\cos(4\,v) \\
&+ 55680\,v^2\,\cos(2\,v) - 1440\,\cos(7\,v) \\
&+ 1440\,\cos(8\,v) - 9600\,v\,\sin(5\,v) \\
&- 7920\,v\,\sin(3\,v) - 3120\,v\,\sin(v) \\
&- 86400\,v^2\,\cos(v)\,, \qquad (13.23) \\
Formula_{37} &= -2880 + 20160\,v\,\sin(4\,v) + 2880\,\cos(6\,v) \\
&- 2880\,\cos(5\,v) + 2880\,\cos(4\,v) \\
&+ 2880\,\cos(v) + 80640\,v^2 + 2880\,\cos(11\,v)
\end{aligned}$$

$$
\begin{aligned}
&+ && 23040\,v^2\,\cos(2\,v) - 3800\,v^3\,\sin(7\,v)\\
&+ && 51015\,v^3\,\sin(5\,v) + 35815\,v^3\,\sin(3\,v)\\
&- && 264613\,v^3\,\sin(v) - 2880\,\cos(7\,v)\\
&- && 23040\,v\,\sin(5\,v) + 28800\,v\,\sin(v)\\
&- && 119520\,v^2\,\cos(v)\,, && (13.24)\\
Formula_{38} &= && -2880 + 24480\,v\,\sin(4\,v) + 2880\,\cos(6\,v)\\
&- && 2880\,\cos(5\,v) + 2880\,\cos(4\,v)\\
&- && 2880\,\cos(3\,v) - 1440\,\cos(2\,v)\\
&+ && 4320\,\cos(v) + 100800\,v^2\\
&- && 2880\,\cos(12\,v) + 9348\,v^3\,\sin(10\,v)\\
&+ && 4320\,v^2\,\cos(12\,v) - 5760\,v\,\sin(12\,v)\\
&+ && 2880\,\cos(11\,v) + 2880\,v\,\sin(11\,v)\\
&- && 720\,v\,\sin(10\,v) - 5073\,v^3\,\sin(8\,v)\\
&- && 21698\,v^3\,\sin(6\,v) - 49438\,v^3\,\sin(4\,v)\\
&+ && 137522\,v^3\,\sin(2\,v) - 1440\,\cos(10\,v)\\
&+ && 1440\,\cos(9\,v) + 10080\,v\,\sin(8\,v)\\
&- && 8640\,v^2\,\cos(5\,v) - 17280\,v^2\,\cos(3\,v)\\
&+ && 17280\,v\,\sin(6\,v) - 16560\,v\,\sin(2\,v)\\
&- && 720\,v\,\sin(9\,v) - 7200\,v\,\sin(7\,v)\\
&- && 8640\,v^2\,\cos(8\,v) - 11520\,v^2\,\cos(6\,v)\\
&- && 10080\,v^2\,\cos(4\,v) + 97920\,v^2\,\cos(2\,v)\\
&- && 2880\,\cos(7\,v) + 2880\,\cos(8\,v)\\
&- && 14400\,v\,\sin(5\,v) - 21600\,v\,\sin(3\,v)\\
&- && 10800\,v\,\sin(v) - 146880\,v^2\,\cos(v)\,, && (13.25)\\
Formula_{39} &= && 4320\,v^3\,\sin(9\,v) - 2880\,v^3\,\sin(7\,v)\\
&- && 21600\,v^3\,\sin(5\,v) + 18720\,v^3\,\sin(3\,v)\\
&+ && 33120\,v^3\,\sin(v)\,, && (13.26)\\
Formula_{40} &= && 4320\,v^3\,\sin(7\,v) - 2880\,v^3\,\sin(3\,v)\\
&- && 11520\,v^3\,\sin(5\,v) + 36000\,v^3\,\sin(v)\,. && (13.27)
\end{aligned}
$$

Appendix H - $Formula_{41}$, and $Formula_{42}$

$$
\begin{aligned}
Formula_{41} &= -112902750\, v + 11542500\, v\, A_2(v) + 32062500\, v\, A_0(v) \\
&+ 5130000\, v\, A_3(v) + 1282500\, v\, A_4(v) \\
&+ 20087657280 \sin(4v)\, v^4 A_3(v) \\
&+ 31104000 \sin(3v)\, v^4 A_2(v)\, A_3(v)\, A_4(v) \\
&+ 3110400000 \cos(6v)\, v^3 A_0(v)\, A_3(v) \\
&- 1312200000 \cos(5v)\, v^5 A_0(v)\, A_2(v)^2 \\
&- 129600000 \cos(5v)\, v^5 A_0(v)\, A_3(v)\, A_4(v) \\
&- 46656000 \cos(3v)\, v^5 A_2(v)\, A_3(v)\, A_4(v) \\
&- 15552000 \cos(2v)\, v\, A_3(v)^2 + 10368000 \cos(v)\, v^3 A_3(v)\, A_4(v)^2 \\
&- 684547200 \sin(3v)\, v^4 A_2(v)\, A_4(v) \\
&+ 259200000 \cos(v)\, v^3 A_0(v)\, A_3(v)\, A_4(v) \\
&- 17113680000 \sin(3v)\, v^4 A_0(v)\, A_2(v) \\
&+ 104976000 \cos(2v)\, v^3 A_2(v)^2 A_3(v) \\
&+ 10497600000 \sin(6v)\, v^4 A_0(v)\, A_2(v) \\
&+ 120525943680 \cos(6v)\, v^3 \\
&- 100438286400 \cos(5v)\, v^3 + 23328000 \sin(v)\, v^4 A_2(v)\, A_4(v)^2 \\
&+ 20087657280 \sin(2v)\, v^4 A_3(v) + 30131485920 \sin(3v)\, v^4 A_2(v) \\
&+ 5021914320 \sin(4v)\, v^4 A_4(v) - 1826172270\, v \cos(4v) \\
&- 90763200 \sin(6v) + 2073600\, v\, A_0(v) \cos(5v) \\
&+ 336441600\, v\, A_2(v) \cos(3v) + 339681600\, v\, A_3(v) \cos(2v) \\
&+ 341625600\, v\, A_4(v) \cos(v) - 1296000 \sin(v)\, A_4(v) \\
&- 38880000\, A_0(v) \sin(5v) - 3888000\, A_2(v) \sin(3v) \\
&- 2592000\, A_3(v) \sin(2v) + 28522800 \sin(4v) \\
&+ 97891200 \sin(5v) + 10043828640 \sin(v)\, v^4 A_4(v) \\
&- 102682080000 \sin(6v)\, v^4 A_0(v) \\
&+ 4107283200 \cos(4v)\, v^5 A_2(v)\, A_3(v) \\
&- 917913600 \cos(v)\, v^3 A_3(v)\, A_4(v) \\
&+ 20736000 \cos(3v)\, v^3 A_2(v)\, A_3(v)^2 \\
&- 6840000\, v^3 A_3(v)^2 - 5184000 \cos(v)\, v^5 A_3(v)\, A_4(v)^2 \\
&- 959385600 \cos(3v)\, v^3 A_2(v)\, A_3(v) \\
&+ 10368000 \cos(5v)\, v^3 A_0(v)\, A_3(v)\, A_4(v) \\
&- 267187500\, v^3 A_0(v)^2 + 23328000 \cos(2v)\, v^3 A_2(v)\, A_3(v)\, A_4(v) \\
&+ 4665600000 \sin(6v)\, v^4 A_0(v)\, A_3(v) + 21392100 \cos(4v)\, v\, A_4(v) \\
&+ 23328000 \cos(3v)\, v^3 A_2(v)^2 A_4(v) - 34627500\, v^3 A_2(v)^2 \\
&+ 419904000 \sin(6v)\, v^4 A_2(v)\, A_4(v) \\
&+ 2430000000 \sin(3v)\, v^4 A_0(v)^2 A_2(v) \\
&+ 93312000 \cos(5v)\, v^3 A_0(v)\, A_2(v)\, A_3(v) \\
&\ 65305810803 \cos(4v)\, v^3
\end{aligned}
$$

$$
\begin{aligned}
- & \quad 41472000 \cos(2v)\, v^5 A_3(v)^3 \\
- & \quad 777600000 \sin(6v)\, v^2 A_0(v)\, A_3(v) + 11664000 \sin(6v)\, A_2(v) \\
+ & \quad 10368000 \cos(3v)\, v^3 A_2(v)\, A_3(v)\, A_4(v) + 75268500\, v^3 A_4(v) \\
- & \quad 472392000 \cos(3v)\, v^5 A_2(v)^3 - 3634329600 \cos(5v)\, v^3 A_0(v)\, A_3(v) \\
- & \quad 10125000000 \cos(5v)\, v^5 A_0(v)^3 - 10952755200 \cos(6v)\, v^3 A_3(v) \\
- & \quad 24643699200 \cos(6v)\, v^3 A_2(v) - 68454720000 \cos(6v)\, v^3 A_0(v) \\
- & \quad 229478400 \cos(v)\, v^3 A_4(v)^2 - 933465600 \cos(2v)\, v^3 A_3(v)^2 \\
- & \quad 2158617600 \cos(3v)\, v^3 A_2(v)^2 - 7130700 \cos(4v)\, v^3 A_4(v)^2 \\
- & \quad 114091200 \cos(4v)\, v^3 A_3(v)^2 - 1166400000 \sin(5v)\, v^4 A_2(v)\, A_3(v) \\
- & \quad 577586700 \cos(4v)\, v^3 A_2(v)^2 - 4456687500 \cos(4v)\, v^3 A_0(v)^2 \\
- & \quad 3645000000 \cos(3v)\, v^5 A_0(v)^2 A_2(v) \\
+ & \quad 3888000 \sin(3v)\, v^4 A_2(v)\, A_4(v)^2 \\
+ & \quad 524880000 \sin(5v)\, v^4 A_0(v)\, A_2(v)^2 \\
+ & \quad 259200000 \sin(v)\, v^4 A_0(v)\, A_3(v)\, A_4(v) \\
- & \quad 4107283200 \sin(6v)\, v^4 A_4(v) - 20736000 \cos(2v)\, v^5 A_3(v)^2 A_4(v) \\
- & \quad 16429132800 \sin(6v)\, v^4 A_3(v) - 36965548800 \sin(6v)\, v^4 A_2(v) \\
- & \quad 228182400 \sin(4v)\, v^4 A_3(v)\, A_4(v) \\
- & \quad 3208815000 \cos(4v)\, v^3 A_0(v)\, A_2(v) \\
- & \quad 933120000 \cos(5v)\, v^3 A_2(v)\, A_3(v) + 192528900 \cos(4v)\, v A_2(v) \\
+ & \quad 186624000 \sin(6v)\, v^4 A_3(v)\, A_4(v) - 3888000 \cos(v)\, v A_4(v)^2 \\
+ & \quad 583200000 \cos(v)\, v^3 A_0(v)\, A_2(v)\, A_4(v) \\
- & \quad 7803648000 \sin(5v)\, v^4 A_0(v)\, A_3(v) \\
- & \quad 2100297600 \cos(2v)\, v^3 A_2(v)\, A_3(v) - 104976000 \cos(3v)\, v^5 A_2(v)^2 A_4(v) \\
- & \quad 15552000 \cos(5v)\, v A_0(v)\, A_3(v) + 583200000 \cos(3v)\, v^3 A_0(v)\, A_2(v)^2 \\
+ & \quad 64800000 \cos(2v)\, v^3 A_0(v)\, A_3(v)\, A_4(v) \\
- & \quad 186624000 \cos(2v)\, v^5 A_2(v)\, A_3(v)^2 \\
- & \quad 21375000\, v^3 A_0(v)\, A_4(v) + 1620000000 \sin(2v)\, v^4 A_0(v)^2 A_3(v) \\
- & \quad 291600000 \cos(v)\, v^5 A_0(v)\, A_2(v)\, A_4(v) \\
+ & \quad 11409120000 \cos(4v)\, v^5 A_0(v)\, A_3(v) \\
- & \quad 192375000\, v^3 A_0(v)\, A_2(v) + 1026820800 \cos(4v)\, v^5 A_2(v)\, A_4(v) \\
- & \quad 1369094400 \cos(6v)\, v + 31104000 \sin(5v)\, v^2 A_3(v)\, A_4(v) \\
- & \quad 125547858000 \sin(5v)\, v^4 + 32983765920 \sin(5v)\, v^2 \\
+ & \quad 534802500 \cos(4v)\, v A_0(v) - 456364800 \sin(2v)\, v^4 A_3(v)\, A_4(v) \\
- & \quad 259200000 \cos(5v)\, v^5 A_0(v)\, A_3(v)^2 + 85568400 \cos(4v)\, v A_3(v) \\
+ & \quad 64800000 \cos(5v)\, v^3 A_0(v)^2 A_4(v) - 3420000\, v^3 A_3(v)\, A_4(v) \\
- & \quad 57045600 \cos(4v)\, v^3 A_3(v)\, A_4(v) + 20736000 \sin(v)\, v^4 A_3(v)^2 A_4(v)
\end{aligned}
$$

$$
\begin{aligned}
&+ 777600000 \sin(3v)\, v^4 A_0(v)\, A_2(v)\, A_3(v) \\
&+ 810000000 \cos(2v)\, v^3 A_0(v)^2 A_3(v) \\
&- 11664000 \sin(5v) A_2(v) + 93312000 \sin(v)\, v^4 A_2(v)\, A_3(v)\, A_4(v) \\
&+ 2852280000 \cos(4v)\, v^5 A_0(v)\, A_4(v) + 456364800 \cos(v)\, v^5 A_3(v)\, A_4(v) \\
&+ 1140912000 \cos(5v)\, v - 97200000 \cos(2v)\, v\, A_0(v)\, A_3(v) \\
&+ 1296000000 \sin(5v)\, v^4 A_0(v)^2 A_3(v) - 34238769120 \sin(6v)\, v^2 \\
&+ 11409120000 \cos(2v)\, v^5 A_0(v)\, A_3(v) - 34992000 \cos(2v)\, v\, A_2(v)\, A_3(v) \\
&- 93312000 \cos(3v)\, v^5 A_2(v)\, A_3(v)^2 + 777600000 \sin(5v)\, v^2 A_0(v)\, A_3(v) \\
&- 7290000000 \cos(5v)\, v^5 A_0(v)^2 A_2(v) + 1296000 \cos(5v)\, v^3 A_0(v)\, A_4(v)^2 \\
&- 1620000000 \cos(2v)\, v^5 A_0(v)^2 A_3(v) + 62208000 \sin(3v)\, v^4 A_2(v)\, A_3(v)^2 \\
&- 419904000 \cos(3v)\, v^5 A_2(v)^2 A_3(v) + 259200000 \cos(2v)\, v^3 A_0(v)\, A_3(v)^2 \\
&- 5832000 \cos(3v)\, v^5 A_2(v)\, A_4(v)^2 + 6480000 \sin(5v)\, v^4 A_0(v)\, A_4(v)^2 \\
&- 912729600 \sin(v)\, v^4 A_3(v)\, A_4(v) - 7695000\, v^3 A_2(v)\, A_4(v) \\
&+ 4107283200 \cos(2v)\, v^5 A_2(v)\, A_3(v) - 239846400 \cos(3v)\, v^3 A_2(v)\, A_4(v) \\
&- 97200000 \cos(5v)\, v\, A_0(v)^2 + 32400000 \sin(6v)\, A_0(v) \\
&+ 93312000 \cos(3v)\, v^3 A_2(v)^2 A_3(v) + 20736000 \sin(2v)\, v^4 A_3(v)^2 A_4(v) \\
&- 291600000 \sin(5v)\, v^4 A_2(v)\, A_4(v) - 356535000 \cos(4v)\, v^3 A_0(v)\, A_4(v) \\
&- 85500000\, v^3 A_0(v)\, A_3(v) - 31104000 \sin(6v)\, v^2 A_3(v)\, A_4(v) \\
&+ 279936000 \cos(6v)\, v^3 A_2(v)\, A_4(v) + 442095857304 \cos(4v)\, v^5 \\
&+ 10368000 \sin(v)\, v^4 A_3(v)\, A_4(v)^2 - 34992000 \cos(v)\, v\, A_2(v)\, A_4(v) \\
&+ 518400000 \sin(2v)\, v^4 A_0(v)\, A_3(v)^2 \\
&- 46656000 \cos(2v)\, v^5 A_2(v)\, A_3(v)\, A_4(v) \\
&- 34992000 \cos(3v)\, v\, A_2(v)^2 - 10368000 \cos(v)\, v^5 A_3(v)^2 A_4(v) \\
&+ 180788915520 \sin(6v)\, v^4 - 221047928652 \sin(4v)\, v^4 \\
&+ 1296000 \cos(2v)\, v^3 A_3(v)\, A_4(v)^2 - 291600000 \cos(5v)\, v^5 A_0(v)\, A_2(v)\, A_4(v) \\
&+ 677416500\, v^3 A_2(v) + 93312000 \cos(v)\, v^3 A_2(v)\, A_3(v)\, A_4(v) \\
&+ 1296000 \cos(3v)\, v^3 A_2(v)\, A_4(v)^2 \\
&+ 259200000 \cos(3v)\, v^3 A_0(v)\, A_2(v)\, A_3(v) \\
&- 209952000 \cos(2v)\, v^5 A_2(v)^2 A_3(v) - 233366400 \cos(2v)\, v^3 A_3(v)\, A_4(v) \\
&+ 1026820800 \cos(3v)\, v^5 A_2(v)\, A_4(v) - 3888000 \cos(3v)\, v\, A_2(v)\, A_4(v) \\
&+ 777600000 \cos(6v)\, v^3 A_0(v)\, A_4(v) + 45197228880 \sin(4v)\, v^4 A_2(v) \\
&+ 1166400000 \sin(2v)\, v^4 A_0(v)\, A_2(v)\, A_3(v) \\
&+ 125547858000 \sin(4v)\, v^4 A_0(v) \\
&+ 2852280000 \sin(5v)\, v^4 A_4(v) - 405000000 \cos(v)\, v^5 A_0(v)^2 A_4(v) \\
&+ 456364800 \cos(2v)\, v^5 A_3(v)\, A_4(v) + 11409120000 \sin(5v)\, v^4 A_3(v) \\
&+ 25670520000 \sin(5v)\, v^4 A_2(v) + 121526143200 \sin(5v)\, v^4 A_0(v) \\
&+ 64800000 \sin(v)\, v^4 A_0(v)\, A_4(v)^2 + 124416000 \cos(6v)\, v^3 A_3(v)\, A_4(v) \\
&- 2065305600 \cos(v)\, v^3 A_2(v)\, A_4(v) + 1749600000 \sin(3v)\, v^4 A_0(v)\, A_2(v)^2 \\
&- 12960000 \cos(5v)\, v^3 A_4(v)^2 - 207360000 \cos(5v)\, v^3 A_3(v)^2
\end{aligned}
$$

$$
\begin{aligned}
&- 1049760000 \cos(5v)\, v^3 A_2(v)^2 - 14614560000 \cos(5v)\, v^3 A_0(v)^2 \\
&+ 15552000 \cos(6v)\, v^3 A_4(v)^2 + 248832000 \cos(6v)\, v^3 A_3(v)^2 \\
&+ 1259712000 \cos(6v)\, v^3 A_2(v)^2 + 9720000000 \cos(6v)\, v^3 A_0(v)^2 \\
&+ 1296000 \cos(v)\, v^3 A_4(v)^3 + 810000000 \cos(v)\, v^3 A_0(v)^2 A_4(v) \\
&+ 2852280000 \cos(5v)\, v^5 A_0(v)\, A_4(v) - 5021914320 \cos(v)\, v^5 A_4(v) \\
&+ 583200000 \sin(v)\, v^4 A_0(v)\, A_2(v)\, A_4(v) \\
&- 2624400000 \cos(3v)\, v^5 A_0(v)\, A_2(v)^2 \\
&+ 20736000 \cos(2v)\, v^3 A_3(v)^3 - 20087657280 \cos(2v)\, v^5 A_3(v) \\
&+ 104976000 \cos(3v)\, v^3 A_2(v)^3 - 45197228880 \cos(3v)\, v^5 A_2(v) \\
&- 10043828640 \cos(4v)\, v^5 A_4(v) - 40175314560 \cos(4v)\, v^5 A_3(v) \\
&- 90394457760 \cos(4v)\, v^5 A_2(v) + 104976000 \sin(v)\, v^4 A_2(v)^2 A_4(v) \\
&- 251095716000 \cos(4v)\, v^5 A_0(v) + 23328000 \sin(6v)\, v^4 A_4(v)^2 \\
&+ 1889568000 \sin(6v)\, v^4 A_2(v)^2 + 373248000 \sin(6v)\, v^4 A_3(v)^2 \\
&- 228182400 \sin(v)\, v^4 A_4(v)^2 + 14580000000 \sin(6v)\, v^4 A_0(v)^2 \\
&- 1825459200 \sin(2v)\, v^4 A_3(v)^2 - 6160924800 \sin(3v)\, v^4 A_2(v)^2 \\
&- 456364800 \sin(4v)\, v^4 A_3(v)^2 - 28522800 \sin(4v)\, v^4 A_4(v)^2 \\
&- 2310346800 \sin(4v)\, v^4 A_2(v)^2 - 17826750000 \sin(4v)\, v^4 A_0(v)^2 \\
&- 16200000 \sin(5v)\, v^4 A_4(v)^2 - 259200000 \sin(5v)\, v^4 A_3(v)^2 \\
&- 1312200000 \sin(5v)\, v^4 A_2(v)^2 - 38647800000 \sin(5v)\, v^4 A_0(v)^2 \\
&- 125547858000 \cos(5v)\, v^5 A_0(v) + 114091200 \cos(v)\, v^5 A_4(v)^2 \\
&+ 810000000 \cos(5v)\, v^3 A_0(v)^3 + 1825459200 \cos(2v)\, v^5 A_3(v)^2 \\
&+ 9241387200 \cos(3v)\, v^5 A_2(v)^2 + 57045600 \cos(4v)\, v^5 A_4(v)^2 \\
&+ 912729600 \cos(4v)\, v^5 A_3(v)^2 + 35653500000 \cos(4v)\, v^5 A_0(v)^2 \\
&+ 4620693600 \cos(4v)\, v^5 A_2(v)^2 + 279936000 \sin(3v)\, v^4 A_2(v)^2 A_3(v) \\
&+ 1296000 \sin(v)\, v^4 A_4(v)^3 + 41472000 \sin(2v)\, v^4 A_3(v)^3 \\
&+ 810000000 \sin(v)\, v^4 A_0(v)^2 A_4(v) + 69984000 \sin(3v)\, v^4 A_2(v)^2 A_4(v) \\
&- 291600000 \cos(3v)\, v^5 A_0(v)\, A_2(v)\, A_4(v) - 30780000\, v^3 A_2(v)\, A_3(v) \\
&+ 69984000 \sin(5v)\, v^2 A_2(v)\, A_4(v) + 25670520000 \cos(3v)\, v^5 A_0(v)\, A_2(v) \\
&+ 314928000 \sin(3v)\, v^4 A_2(v)^3 + 71307000000 \cos(5v)\, v^5 A_0(v)^2 \\
&+ 4050000000 \sin(5v)\, v^4 A_0(v)^3 - 52488000 \cos(v)\, v^5 A_2(v)^2 A_4(v) \\
&- 648000 \cos(v)\, v^5 A_4(v)^3 - 12960000 \cos(5v)\, v\, A_4(v) \\
&+ 194400000 \sin(3v)\, v^4 A_0(v)\, A_2(v)\, A_4(v) \\
&+ 2592000 \sin(2v)\, v^4 A_3(v)\, A_4(v)^2 \\
&- 116640000 \cos(5v)\, v\, A_2(v) + 15552000 \cos(6v)\, v\, A_4(v) \\
&- 51840000 \cos(5v)\, v\, A_3(v) - 129600000 \sin(5v)\, v^4 A_3(v)\, A_4(v) \\
&+ 62208000 \cos(6v)\, v\, A_3(v) + 139968000 \cos(6v)\, v\, A_2(v)
\end{aligned}
$$

$$
\begin{aligned}
&+ 388800000 \cos(6v)\, v\, A_0(v) + 731203200 \sin(6v)\, v^2 A_4(v) \\
&+ 6580828800 \sin(6v)\, v^2 A_2(v) + 2924812800 \sin(6v)\, v^2 A_3(v) \\
&+ 18280080000 \sin(6v)\, v^2 A_0(v) - 2867788800 \sin(5v)\, v^2 A_3(v) \\
&- 716947200 \sin(5v)\, v^2 A_4(v) - 6452524800 \sin(5v)\, v^2 A_2(v) \\
&+ 10500193440 \cos(2v)\, v^3 A_3(v) - 17923680000 \sin(5v)\, v^2 A_0(v) \\
&+ 10157919840 \cos(v)\, v^3 A_4(v) + 1369569780 \cos(4v)\, v^3 A_4(v) \\
&+ 11070649440 \cos(3v)\, v^3 A_2(v) + 12326128020 \cos(4v)\, v^3 A_2(v) \\
&+ 5478279120 \cos(4v)\, v^3 A_3(v) - 518400000 \cos(2v)\, v^5 A_0(v)\, A_3(v)^2 \\
&- 62208000 \sin(6v)\, v^2 A_3(v)^2 - 3888000 \sin(6v)\, v^2 A_4(v)^2 \\
&+ 34239244500 \cos(4v)\, v^3 A_0(v) - 314928000 \sin(6v)\, v^2 A_2(v)^2 \\
&+ 314928000 \sin(5v)\, v^2 A_2(v)^2 + 62208000 \sin(5v)\, v^2 A_3(v)^2 \\
&+ 3888000 \sin(5v)\, v^2 A_4(v)^2 - 2430000000 \sin(6v)\, v^2 A_0(v)^2 \\
&+ 2430000000 \sin(5v)\, v^2 A_0(v)^2 + 2281824000 \cos(5v)\, v^3 A_4(v) \\
&+ 69941708640 \cos(5v)\, v^3 A_0(v) + 9127296000 \cos(5v)\, v^3 A_3(v) \\
&+ 20536416000 \cos(5v)\, v^3 A_2(v) - 2738188800 \cos(6v)\, v^3 A_4(v) \\
&- 1749600000 \sin(6v)\, v^2 A_0(v)\, A_2(v) - 3313068475\, v^3 \\
&+ 186624000 \sin(2v)\, v^4 A_2(v)\, A_3(v)^2 \\
&- 17558208000 \sin(5v)\, v^4 A_0(v)\, A_2(v) \\
&- 5704560000 \sin(4v)\, v^4 A_0(v)\, A_3(v) \\
&- 5704560000 \sin(v)\, v^4 A_0(v)\, A_4(v) \\
&- 1426140000 \cos(4v)\, v^3 A_0(v)\, A_3(v) \\
&+ 1296000 \sin(6v)\, A_4(v) - 233280000 \cos(5v)\, v^3 A_2(v)\, A_4(v) \\
&+ 1749600000 \sin(5v)\, v^2 A_0(v)\, A_2(v) \\
&+ 93312000 \cos(2v)\, v^3 A_2(v)\, A_3(v)^2 \\
&+ 209952000 \sin(2v)\, v^4 A_2(v)^2 A_3(v) \\
&- 2738188800 \sin(3v)\, v^4 A_2(v)\, A_3(v) \\
&- 15552000 \cos(3v)\, v\, A_2(v)\, A_3(v) \\
&- 12835260000 \sin(4v)\, v^4 A_0(v)\, A_2(v) \\
&+ 456364800 \cos(4v)\, v^5 A_3(v)\, A_4(v) \\
&+ 64800000 \cos(3v)\, v^3 A_0(v)\, A_2(v)\, A_4(v) \\
&+ 23328000 \cos(v)\, v^3 A_2(v)\, A_4(v)^2 \\
&+ 116640000 \sin(5v)\, v^4 A_0(v)\, A_2(v)\, A_4(v) \\
&+ 583200000 \cos(5v)\, v^3 A_0(v)^2 A_2(v) - 97200000 \cos(v)\, v\, A_0(v)\, A_4(v) \\
&- 129600000 \cos(v)\, v^5 A_0(v)\, A_3(v)\, A_4(v) \\
&+ 129600000 \sin(2v)\, v^4 A_0(v)\, A_3(v)\, A_4(v) \\
&- 3888000 \cos(5v)\, v\, A_0(v)\, A_4(v) - 810000000 \cos(5v)\, v^5 A_0(v)^2 A_4(v) \\
&+ 51840000 \sin(5v)\, v^4 A_0(v)\, A_3(v)\, A_4(v) \\
&- 5834160000 \cos(2v)\, v^3 A_0(v)\, A_3(v) \\
&- 2053641600 \sin(v)\, v^4 A_2(v)\, A_4(v) - 908582400 \cos(5v)\, v^3 A_0(v)\, A_4(v) \\
&+ 20736000 \cos(v)\, v^3 A(v)^2 A(v) + 64800000 \cos(v)\, v^3 A(v)\, A(v)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ && 4107283200 \cos(3v)\ v^5 A_2(v)\ A_3(v) \\
&+ && 279936000 \sin(5v)\ v^2 A_2(v)\ A_3(v) \\
&+ && 23328000 \cos(5v)\ v^3 A_0(v)\ A_2(v)\ A_4(v) \\
&- && 34992000 \cos(5v)\, v\, A_0(v)\ A_2(v) \\
&+ && 1119744000 \cos(6v)\ v^3 A_2(v)\ A_3(v) \\
&- && 194400000 \sin(6v)\ v^2 A_0(v)\ A_4(v) \\
&+ && 46656000 \sin(2v)\ v^4 A_2(v)\ A_3(v)\ A_4(v) \\
&- && 129600000 \cos(2v)\ v^5 A_0(v)\ A_3(v)\ A_4(v) \\
&- && 2053641600 \sin(4v)\ v^4 A_2(v)\ A_3(v) - 1296000 \sin(5v)\, A_4(v) \\
&- && 4107283200 \sin(2v)\ v^4 A_2(v)\ A_3(v) - 5184000 \sin(5v)\, A_3(v) \\
&- && 279936000 \sin(6v)\ v^2 A_2(v)\ A_3(v) \\
&- && 3888000 \cos(2v)\, v\, A_3(v)\ A_4(v) \\
&- && 128352600 \cos(4v)\ v^3 A_2(v)\ A_4(v) \\
&+ && 2916000000 \sin(5v)\ v^4 A_0(v)^2 A_2(v) \\
&- && 11409120000 \sin(2v)\ v^4 A_0(v)\ A_3(v) \\
&+ && 6998400000 \cos(6v)\ v^3 A_0(v)\ A_2(v) \\
&- && 103680000 \cos(5v)\ v^3 A_3(v)\ A_4(v) \\
&- && 5996160000 \cos(3v)\ v^3 A_0(v)\ A_2(v) \\
&+ && 1166400000 \sin(6v)\ v^4 A_0(v)\ A_4(v) + 301074000\, v^3 A_3(v) \\
&- && 69984000 \sin(6v)\ v^2 A_2(v)\ A_4(v) \\
&- && 11664000 \cos(v)\ v^5 A_2(v)\ A_4(v)^2 \\
&- && 16200000 \cos(5v)\ v^5 A_0(v)\ A_4(v)^2 \\
&+ && 104976000 \cos(5v)\ v^3 A_0(v)\ A_2(v)^2 \\
&- && 5736960000 \cos(v)\ v^3 A_0(v)\ A_4(v) \\
&+ && 583200000 \cos(2v)\ v^3 A_0(v)\ A_2(v)\ A_3(v) \\
&- && 1426140000 \sin(4v)\ v^4 A_0(v)\ A_4(v) \\
&- && 3240000000 \cos(5v)\ v^5 A_0(v)^2 A_3(v) \\
&- && 1166400000 \cos(2v)\ v^5 A_0(v)\ A_2(v)\ A_3(v) \\
&- && 2592000 \cos(2v)\ v^5 A_3(v)\ A_4(v)^2 \\
&- && 513410400 \cos(4v)\ v^3 A_2(v)\ A_3(v) \\
&+ && 259200000 \cos(5v)\ v^3 A_0(v)^2 A_3(v)\,, \qquad (13.28) \\
Formula_{42} &= && 10125000000\, v^6 A_0(v)^3 + 10935000000\, v^6 A_0(v)^2 A_2(v) \\
&+ && 4860000000\, v^6 A_0(v)^2 A_3(v) + 1215000000\, v^6 A_0(v)^2 A_4(v) \\
&+ && 3936600000\, v^6 A_0(v)\ A_2(v)^2 + 3499200000\, v^6 A_0(v)\ A_2(v)\ A_3(v) \\
&+ && 874800000\, v^6 A_0(v)\ A_2(v)\ A_4(v) + 777600000\, v^6 A_0(v)\ A_3(v)^2 \\
&+ && 388800000\, v^6 A_0(v)\ A_3(v)\ A_4(v) + 48600000\, v^6 A_0(v)\ A_4(v)^2 \\
&+ && 472392000\, v^6 A_2(v)^3 + 629856000\, v^6 A_2(v)^2 A_3(v)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 157464000\, v^6 A_2(v)^2 A_4(v) + 279936000\, v^6 A_2(v)\, A_3(v)^2 \\
&+ \quad 139968000\, v^6 A_2(v)\, A_3(v)\, A_4(v) + 17496000\, v^6 A_2(v)\, A_4(v)^2 \\
&+ \quad 41472000\, v^6 A_3(v)^3 + 31104000\, v^6 A_3(v)^2 A_4(v) \\
&+ \quad 7776000\, v^6 A_3(v)\, A_4(v)^2 + 648000\, v^6 A_4(v)^3 \\
&- \quad 106960500000\, v^6 A_0(v)^2 - 77011560000\, v^6 A_0(v)\, A_2(v) \\
&- \quad 34227360000\, v^6 A_0(v)\, A_3(v) - 8556840000\, v^6 A_0(v)\, A_4(v) \\
&- \quad 13862080800\, v^6 A_2(v)^2 - 12321849600\, v^6 A_2(v)\, A_3(v) \\
&- \quad 3080462400\, v^6 A_2(v)\, A_4(v) - 2738188800\, v^6 A_3(v)^2 \\
&- \quad 1369094400\, v^6 A_3(v)\, A_4(v) - 171136800\, v^6 A_4(v)^2 \\
&+ \quad 376643574000\, v^6 A_0(v) + 135591686640\, v^6 A_2(v) \\
&+ \quad 60262971840\, v^6 A_3(v) + 15065742960\, v^6 A_4(v) - 442095857304\, v^6 \\
&+ \quad 1215000000\, v^4 A_0(v)^2 + 874800000\, v^4 A_0(v)\, A_2(v) \\
&+ \quad 388800000\, v^4 A_0(v)\, A_3(v) + 97200000\, v^4 A_0(v)\, A_4(v) \\
&+ \quad 157464000\, v^4 A_2(v)^2 + 139968000\, v^4 A_2(v)\, A_3(v) \\
&+ \quad 34992000\, v^4 A_2(v)\, A_4(v) + 31104000\, v^4 A_3(v)^2 \\
&+ \quad 15552000\, v^4 A_3(v)\, A_4(v) + 1944000\, v^4 A_4(v)^2 \\
&- \quad 8556840000\, v^4 A_0(v) - 3080462400\, v^4 A_2(v) \\
&- \quad 1369094400\, v^4 A_3(v) - 342273600\, v^4 A_4(v) \\
&+ \quad 15065742960\, v^4 + 48600000\, v^2 A_0(v) \\
&+ \quad 17496000\, v^2 A_2(v) + 7776000\, v^2 A_3(v) \\
&+ \quad 1944000\, v^2 A_4(v) - 171136800\, v^2 + 648000. \qquad (13.29)
\end{aligned}
$$

Appendix I - $Formula_j,\ j = 43\,(1)\,48$

$$
\begin{aligned}
Formula_{43} &= 1440\,v\,\sin(4\,v) + 1440\,\cos(2\,v)\\
&- 1440\,\cos(v) + 1440\,v^2 + 8640\,v^2\,\cos(10\,v)\\
&- 5040\,v\,\sin(10\,v) - 7923\,v^3\,\sin(8\,v) + 8398\,v^3\,\sin(6\,v)\\
&- 30742\,v^3\,\sin(4\,v) + 67982\,v^3\,\sin(2\,v)\\
&- 1440\,\cos(10\,v) + 1440\,\cos(9\,v)\\
&+ 1440\,v\,\sin(8\,v) - 4320\,v^2\,\cos(9\,v)\\
&+ 2880\,v^2\,\cos(7\,v) - 10080\,v^2\,\cos(5\,v)\\
&+ 12960\,v^2\,\cos(3\,v) - 4320\,v\,\sin(6\,v)\\
&+ 12240\,v\,\sin(2\,v) + 3600\,v\,\sin(9\,v)\\
&- 1440\,v\,\sin(7\,v) - 4320\,v^2\,\cos(8\,v)\\
&+ 14400\,v^2\,\cos(6\,v) - 14400\,v^2\,\cos(4\,v)\\
&- 40320\,v^2\,\cos(2\,v) + 4320\,v\,\sin(5\,v)\\
&- 1440\,v\,\sin(3\,v) - 10800\,v\,\sin(v)\\
&+ 27360\,v^2\,\cos(v)\,, \qquad (13.30)\\
Formula_{44} &= 30240\,v^3\,\sin(v) + 2880\,v^3\,\sin(7\,v)\\
&- 10080\,v^3\,\sin(5\,v) + 12960\,v^3\,\sin(3\,v)\\
&- 4320\,v^3\,\sin(9\,v)\,, \qquad (13.31)\\
Formula_{45} &= -4320 + 15120\,v\,\sin(4\,v) + 1440\,\cos(4\,v)\\
&- 1440\,\cos(3\,v) - 2880\,\cos(2\,v)\\
&+ 7200\,\cos(v) + 133920\,v^2 + 4320\,\cos(12\,v)\\
&- 23769\,v^3\,\sin(10\,v) - 8640\,v^2\,\cos(12\,v)\\
&+ 10800\,v\,\sin(12\,v) - 4320\,\cos(11\,v)\\
&- 6480\,v\,\sin(11\,v) - 2880\,v^2\,\cos(10\,v)\\
&+ 4320\,v\,\sin(10\,v) - 2850\,v^3\,\sin(8\,v)\\
&- 4275\,v^3\,\sin(6\,v) - 87476\,v^3\,\sin(4\,v)\\
&+ 318022\,v^3\,\sin(2\,v) + 2880\,\cos(10\,v)\\
&- 2880\,\cos(9\,v) - 2160\,v\,\sin(8\,v)\\
&+ 17280\,v^2\,\cos(3\,v) + 8640\,v\,\sin(6\,v)\\
&- 30240\,v\,\sin(2\,v) - 1440\,v\,\sin(9\,v)\\
&+ 720\,v\,\sin(7\,v) + 1440\,v^2\,\cos(8\,v)\\
&- 8640\,v^2\,\cos(6\,v) - 23040\,v^2\,\cos(4\,v)\\
&+ 115200\,v^2\,\cos(2\,v) + 1440\,\cos(7\,v)\\
&- 1440\,\cos(8\,v) - 8640\,v\,\sin(5\,v) - 13680\,v\,\sin(3\,v)\\
&- 9360\,v\,\sin(v) - 190080\,v^2\,\cos(v)\,, \qquad (13.32)\\
Formula_{46} &= -2880 + 20160\,v\,\sin(4\,v) - 2880\,\cos(6\,v)\\
&- 2880\,\cos(5\,v) + 2880\,\cos(4\,v)\\
&+ 2880\,\cos(v) + 80640\,v^2 - 2880\,\cos(11\,v)\\
&+ 4320\,v^2\,\cos(11\,v) - 5760\,v\,\sin(11\,v)
\end{aligned}
$$

$$
\begin{aligned}
& - 42465\, v^3 \sin(5\, v) + 170183\, v^3 \sin(3\, v) \\
& - 396131\, v^3 \sin(v) + 2880 \cos(7\, v) \\
& - 23040\, v \sin(5\, v) + 28800\, v \sin(v) \\
& - 96480\, v^2 \cos(v)\,, \qquad (13.33)
\end{aligned}
$$

$$
\begin{aligned}
Formula_{47} = {} & 11520\, v^3 \sin(5\, v) - 28800\, v^3 \sin(3\, v) \\
& + 59040\, v^3 \sin(v) - 4320\, v^3 \sin(7\, v)\,, \qquad (13.34)
\end{aligned}
$$

$$
\begin{aligned}
Formula_{48} = {} & -2880 + 24480\, v \sin(4\, v) + 2880 \cos(4\, v) \\
& - 2880 \cos(3\, v) - 1440 \cos(2\, v) \\
& + 4320 \cos(v) + 83520\, v^2 + 2880 \cos(12\, v) \\
& - 9348\, v^3 \sin(10\, v) - 4320\, v^2 \cos(12\, v) \\
& + 5760\, v \sin(12\, v) - 2880 \cos(11\, v) \\
& - 2880\, v \sin(11\, v) + 720\, v \sin(10\, v) \\
& + 5073\, v^3 \sin(8\, v) + 2698\, v^3 \sin(6\, v) \\
& - 56734\, v^3 \sin(4\, v) + 131822\, v^3 \sin(2\, v) \\
& + 1440 \cos(10\, v) - 1440 \cos(9\, v) \\
& - 10080\, v \sin(8\, v) + 8640\, v^2 \cos(5\, v) \\
& + 17280\, v^2 \cos(3\, v) + 7200\, v \sin(6\, v) \\
& - 16560\, v \sin(2\, v) + 720\, v \sin(9\, v) \\
& + 7200\, v \sin(7\, v) + 8640\, v^2 \cos(8\, v) \\
& - 11520\, v^2 \cos(6\, v) - 36000\, v^2 \cos(4\, v) \\
& + 63360\, v^2 \cos(2\, v) + 2880 \cos(7\, v) \\
& - 2880 \cos(8\, v) - 7200\, v \sin(5\, v) \\
& - 21600\, v \sin(3\, v) - 10800\, v \sin(v) \\
& - 112320\, v^2 \cos(v)\,. \qquad (13.35)
\end{aligned}
$$

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